Basic Linear Algebra

In this chapter, we study the topics in linear algebra that will be needed in the rest of the book. We begin by discussing the building blocks of linear algebra: matrices and vectors. Then we use our knowledge of matrices and vectors to develop a systematic procedure (the Gauss–Jordan method) for solving linear equations, which we then use to invert matrices. We close the chapter with an introduction to determinants.

The material covered in this chapter will be used in our study of linear and nonlinear programming.

2.1 Matrices and Vectors

Matrices

DEFINITION ■ A **matrix** is any rectangular array of numbers. ■

For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \end{bmatrix}$$

are all matrices.

If a matrix A has m rows and n columns, we call A an $m \times n$ matrix. We refer to $m \times n$ as the **order** of the matrix. A typical $m \times n$ matrix A may be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

DEFINITION ■ The number in the ith row and jth column of A is called the ***ij*th element** of A and is written a_{ij}. ■

For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

then $a_{11} = 1$, $a_{23} = 6$, and $a_{31} = 7$.

Sometimes we will use the notation $A = [a_{ij}]$ to indicate that A is the matrix whose ijth element is a_{ij}.

DEFINITION ■ Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** if and only if A and B are of the same order and for all i and j, $a_{ij} = b_{ij}$. ■

For example, if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} x & y \\ w & z \end{bmatrix}$$

then $A = B$ if and only if $x = 1$, $y = 2$, $w = 3$, and $z = 4$.

Vectors

Any matrix with only one column (that is, any $m \times 1$ matrix) may be thought of as a **column vector.** The number of rows in a column vector is the **dimension** of the column vector. Thus,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

may be thought of as a 2×1 matrix or a two-dimensional column vector. R^m will denote the set of all m-dimensional column vectors.

In analogous fashion, we can think of any vector with only one row (a $1 \times n$ matrix as a **row vector.** The dimension of a row vector is the number of columns in the vector. Thus, $[9 \quad 2 \quad 3]$ may be viewed as a 1×3 matrix or a three-dimensional row vector. In this book, vectors appear in boldface type: for instance, vector **v.** An m-dimensional vector (either row or column) in which all elements equal zero is called a **zero vector** (written **0**). Thus,

$$[0 \quad 0] \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

are two-dimensional zero vectors.

Any m-dimensional vector corresponds to a directed line segment in the m-dimensional plane. For example, in the two-dimensional plane, the vector

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

corresponds to the line segment joining the point

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

to the point

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The directed line segments corresponding to

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

are drawn in Figure 1.

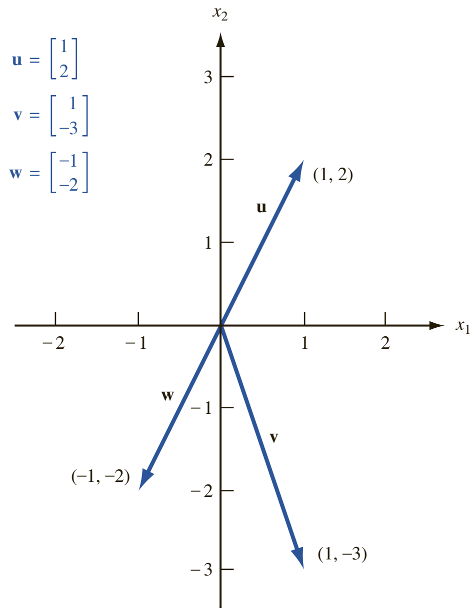

FIGURE 1
Vectors Are Directed Line Segments

The Scalar Product of Two Vectors

An important result of multiplying two vectors is the *scalar product.* To define the scalar product of two vectors, suppose we have a row vector $\mathbf{u} = [u_1 \quad u_2 \quad \cdots \quad u_n]$ and a column vector

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

of the same dimension. The **scalar product** of **u** and **v** (written $\mathbf{u} \cdot \mathbf{v}$) is the number $u_1v_1 + u_2v_2 + \cdots + u_nv_n$.

For the scalar product of two vectors to be defined, the first vector must be a row vector and the second vector must be a column vector. For example, if

$$\mathbf{u} = [1 \quad 2 \quad 3] \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

then $\mathbf{u} \cdot \mathbf{v} = 1(2) + 2(1) + 3(2) = 10$. By these rules for computing a scalar product, if

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = [2 \quad 3]$$

then $\mathbf{u} \cdot \mathbf{v}$ is not defined. Also, if

$$\mathbf{u} = [1 \quad 2 \quad 3] \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

then $\mathbf{u} \cdot \mathbf{v}$ is not defined because the vectors are of two different dimensions.

Note that two vectors are perpendicular if and only if their scalar product equals 0. Thus, the vectors $[1 \quad -1]$ and $[1 \quad 1]$ are perpendicular.

We note that $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta$, where $\|\mathbf{u}\|$ is the length of the vector **u** and θ is the angle between the vectors **u** and **v**.

Matrix Operations

We now describe the arithmetic operations on matrices that are used later in this book.

The Scalar Multiple of a Matrix

Given any matrix A and any number c (a *number* is sometimes referred to as a *scalar*), the matrix cA is obtained from the matrix A by multiplying each element of A by c. For example,

$$\text{if} \qquad A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \qquad \text{then} \qquad 3A = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

For $c = -1$, scalar multiplication of the matrix A is sometimes written as $-A$.

Addition of Two Matrices

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices with the same order (say, $m \times n$). Then the matrix $C = A + B$ is defined to be the $m \times n$ matrix whose ijth element is $a_{ij} + b_{ij}$. Thus, to obtain the sum of two matrices A and B, we add the corresponding elements of A and B. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -1 & -2 & -3 \\ 2 & 1 & -1 \end{bmatrix}$$

then

$$A + B = \begin{bmatrix} 1-1 & 2-2 & 3-3 \\ 0+2 & -1+1 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

This rule for matrix addition may be used to add vectors of the same dimension. For example, if $\mathbf{u} = [1 \quad 2]$ and $\mathbf{v} = [2 \quad 1]$, then $\mathbf{u} + \mathbf{v} = [1 + 2 \quad 2 + 1] = [3 \quad 3]$. Vectors may be added geometrically by the parallelogram law (see Figure 2).

We can use scalar multiplication and the addition of matrices to define the concept of a line segment. A glance at Figure 1 should convince you that any point u in the m-dimensional plane corresponds to the m-dimensional vector $\mathbf{u}$ formed by joining the origin to the point u. For any two points u and v in the m-dimensional plane, the **line segment** joining u and v (called the line segment uv) is the set of all points in the m-dimensional plane that correspond to the vectors $c\mathbf{u} + (1 - c)\mathbf{v}$, where $0 \leq c \leq 1$ (Figure 3). For example, if $u = (1, 2)$ and $v = (2, 1)$, then the line segment uv consists

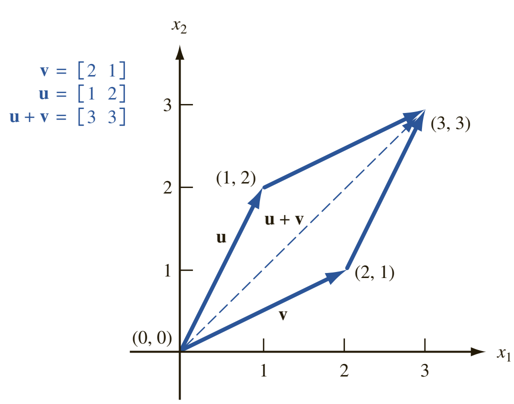

FIGURE 2
Addition of Vectors

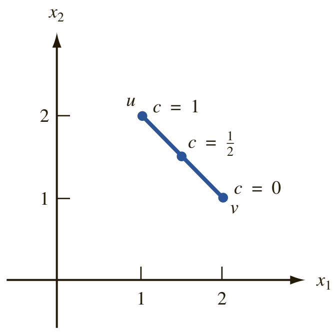

FIGURE 3
Line Segment Joining $u = (1, 2)$ and $v = (2, 1)$

of the points corresponding to the vectors $c[1 \quad 2] + (1 - c)[2 \quad 1] = [2 - c \quad 1 + c]$, where $0 \le c \le 1$. For $c = 0$ and $c = 1$, we obtain the endpoints of the line segment uv; for $c = \frac{1}{2}$, we obtain the midpoint $(0.5\mathbf{u} + 0.5\mathbf{v})$ of the line segment uv.

Using the parallelogram law, the line segment uv may also be viewed as the points corresponding to the vectors $\mathbf{u} + c(\mathbf{v} - \mathbf{u})$, where $0 \le c \le 1$ (Figure 4). Observe that for $c = 0$, we obtain the vector $\mathbf{u}$ (corresponding to point u), and for $c = 1$, we obtain the vector $\mathbf{v}$ (corresponding to point v).

The Transpose of a Matrix

Given any $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

the **transpose** of A (written A^T) is the $n \times m$ matrix

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

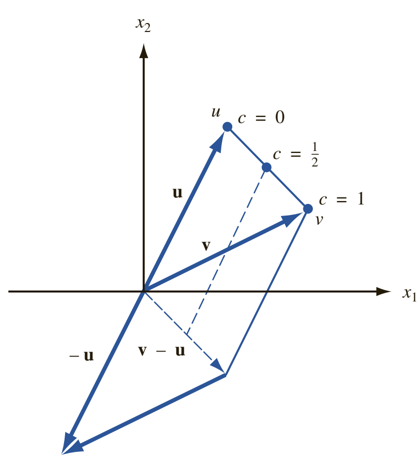

FIGURE 4
Representation of Line Segment uv

Thus, A^T is obtained from A by letting row 1 of A be column 1 of A^T, letting row 2 of A be column 2 of A^T, and so on. For example,

$$\text{if} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \text{then} \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Observe that $(A^T)^T = A$. Let $B = [1 \quad 2]$; then

$$B^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad (B^T)^T = [1 \quad 2] = B$$

As indicated by these two examples, for any matrix A, $(A^T)^T = A$.

Matrix Multiplication

Given two matrices A and B, the matrix product of A and B (written AB) is defined if and only if

$$\text{Number of columns in } A = \text{number of rows in } B \tag{1}$$

For the moment, assume that for some positive integer r, A has r columns and B has r rows. Then for some m and n, A is an $m \times r$ matrix and B is an $r \times n$ matrix.

DEFINITION ■ The **matrix product** $C = AB$ of A and B is the $m \times n$ matrix C whose ijth element is determined as follows:

$$ij\text{th element of } C = \text{scalar product of row } i \text{ of } A \times \text{column } j \text{ of } B \quad ■ \tag{2}$$

If Equation (1) is satisfied, then each row of A and each column of B will have the same number of elements. Also, if (1) is satisfied, then the scalar product in Equation (2) will be defined. The product matrix $C = AB$ will have the same number of rows as A and the same number of columns as B.

EXAMPLE 1 Matrix Multiplication

Compute $C = AB$ for

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$$

Solution Because A is a 2×3 matrix and B is a 3×2 matrix, AB is defined, and C will be a 2×2 matrix. From Equation (2),

$$c_{11} = [1 \quad 1 \quad 2] \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1(1) + 1(2) + 2(1) = 5$$

$$c_{12} = [1 \quad 1 \quad 2] \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 1(1) + 1(3) + 2(2) = 8$$

$$c_{21} = [2 \quad 1 \quad 3] \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 2(1) + 1(2) + 3(1) = 7$$

$$c_{22} = [2 \quad 1 \quad 3]\begin{bmatrix}1\\3\\2\end{bmatrix} = 2(1) + 1(3) + 3(2) = 11$$

$$C = AB = \begin{bmatrix}5 & 8\\7 & 11\end{bmatrix}$$

EXAMPLE 2 Column Vector Times Row Vector

Find AB for

$$A = \begin{bmatrix}3\\4\end{bmatrix} \quad \text{and} \quad B = [1 \quad 2]$$

Solution Because A has one column and B has one row, $C = AB$ will exist. From Equation (2), we know that C is a 2×2 matrix with

$$c_{11} = 3(1) = 3 \qquad c_{21} = 4(1) = 4$$
$$c_{12} = 3(2) = 6 \qquad c_{22} = 4(2) = 8$$

Thus,

$$C = \begin{bmatrix}3 & 6\\4 & 8\end{bmatrix}$$

EXAMPLE 3 Row Vector Times Column Vector

Compute $D = BA$ for the A and B of Example 2.

Solution In this case, D will be a 1×1 matrix (or a scalar). From Equation (2),

$$d_{11} = [1 \quad 2]\begin{bmatrix}3\\4\end{bmatrix} = 1(3) + 2(4) = 11$$

Thus, $D = [11]$. In this example, matrix multiplication is equivalent to scalar multiplication of a row and column vector.

Recall that if you multiply two real numbers a and b, then $ab = ba$. This is called the *commutative property of multiplication.* Examples 2 and 3 show that for matrix multiplication, it may be that $AB \neq BA$. Matrix multiplication is not necessarily commutative. (In some cases, however, $AB = BA$ will hold.)

EXAMPLE 4 Undefined Matrix Product

Show that AB is undefined if

$$A = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}1 & 1\\0 & 1\\1 & 2\end{bmatrix}$$

Solution This follows because A has two columns and B has three rows. Thus, Equation (1) is not satisfied.

TABLE 1

Gallons of Crude Oil Required to Produce 1 Gallon of Gasoline

Crude Oil	Premium Unleaded	Regular Unleaded	Regular Leaded
1	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{4}$
2	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{3}{4}$

Many computations that commonly occur in operations research (and other branches of mathematics) can be concisely expressed by using matrix multiplication.To illustrate this, suppose an oil company manufactures three types of gasoline: premium unleaded, regular unleaded, and regular leaded. These gasolines are produced by mixing two types of crude oil: crude oil 1 and crude oil 2. The number of gallons of crude oil required to manufacture 1 gallon of gasoline is given in Table 1.

From this information, we can find the amount of each type of crude oil needed to manufacture a given amount of gasoline. For example, if the company wants to produce 10 gallons of premium unleaded, 6 gallons of regular unleaded, and 5 gallons of regular leaded, then the company's crude oil requirements would be

$$\text{Crude 1 required} = (\tfrac{3}{4})(10) + (\tfrac{2}{3})(6) + (\tfrac{1}{4})5 = 12.75 \text{ gallons}$$
$$\text{Crude 2 required} = (\tfrac{1}{4})(10) + (\tfrac{1}{3})(6) + (\tfrac{3}{4})5 = 8.25 \text{ gallons}$$

More generally, we define

$$\begin{aligned}
p_U &= \text{gallons of premium unleaded produced}\\
r_U &= \text{gallons of regular unleaded produced}\\
r_L &= \text{gallons of regular leaded produced}\\
c_1 &= \text{gallons of crude 1 required}\\
c_2 &= \text{gallons of crude 2 required}
\end{aligned}$$

Then the relationship between these variables may be expressed by

$$\begin{aligned}
c_1 &= (\tfrac{3}{4})\,p_U + (\tfrac{2}{3})\,r_U + (\tfrac{1}{4})\,r_L\\
c_2 &= (\tfrac{1}{4})\,p_U + (\tfrac{1}{3})\,r_U + (\tfrac{3}{4})\,r_L
\end{aligned}$$

Using matrix multiplication, these relationships may be expressed by

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{2}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} p_U \\ r_U \\ r_L \end{bmatrix}$$

Properties of Matrix Multiplication

To close this section, we discuss some important properties of matrix multiplication. In what follows, we assume that all matrix products are defined.

1 Row i of AB = (row i of A)B. To illustrate this property, let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$$

Then row 2 of the 2×2 matrix AB is equal to

$$[2 \quad 1 \quad 3]\begin{bmatrix}1 & 1\\ 2 & 3\\ 1 & 2\end{bmatrix} = [7 \quad 11]$$

This answer agrees with Example 1.

2 Column j of $AB = A$(column j of B). Thus, for A and B as given, the first column of AB is

$$\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 3\end{bmatrix}\begin{bmatrix}1\\ 2\\ 1\end{bmatrix} = \begin{bmatrix}5\\ 7\end{bmatrix}$$

Properties 1 and 2 are helpful when you need to compute only *part* of the matrix AB.

3 Matrix multiplication is associative. That is, $A(BC) = (AB)C$. To illustrate, let

$$A = [1 \quad 2], \qquad B = \begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix}, \qquad C = \begin{bmatrix}2\\ 1\end{bmatrix}$$

Then $AB = [10 \quad 13]$ and $(AB)C = 10(2) + 13(1) = [33]$.

On the other hand,

$$BC = \begin{bmatrix}7\\ 13\end{bmatrix}$$

so $A(BC) = 1(7) + 2(13) = [33]$. In this case, $A(BC) = (AB)C$ does hold.

4 Matrix multiplication is distributive. That is, $A(B + C) = AB + AC$ and $(B + C)D = BD + CD$.

Matrix Multiplication with Excel

Mmult.xls

Using the Excel MMULT function, it is easy to multiply matrices. To illustrate, let's use Excel to find the matrix product AB that we found in Example 1 (see Figure 5 and file Mmult.xls). We proceed as follows:

Step 1 Enter A and B in D2:F3 and D5:E7, respectively.

Step 2 Select the range (D9:E10) in which the product AB will be computed.

Step 3 In the upper left-hand corner (D9) of the selected range, type the formula

= MMULT(D2:F3,D5:E7)

Then hit **Control Shift Enter** (not just Enter), and the desired matrix product will be computed. Note that MMULT is an *array* function and not an ordinary spreadsheet function. This explains why we must preselect the range for AB and use Control Shift Enter.

	A	B	C	D	E	F
1	MatrixMultiplication					
2				1	1	2
3			A	2	1	3
4						
5			B	1	1	
6				2	3	
7				1	2	
8						
9				5	8	
10			C	7	11	
11						

FIGURE 5

PROBLEMS

Group A

1 For $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 2 \end{bmatrix}$, find:

a $-A$ **b** $3A$ **c** $A + 2B$
d A^T **e** B^T **f** AB
g BA

2 Only three brands of beer (beer 1, beer 2, and beer 3) are available for sale in Metropolis. From time to time, people try one or another of these brands. Suppose that at the beginning of each month, people change the beer they are drinking according to the following rules:

30% of the people who prefer beer 1 switch to beer 2.
20% of the people who prefer beer 1 switch to beer 3.
30% of the people who prefer beer 2 switch to beer 3.
30% of the people who prefer beer 3 switch to beer 2.
10% of the people who prefer beer 3 switch to beer 1.

For $i = 1, 2, 3$, let x_i be the number who prefer beer i at the beginning of this month and y_i be the number who prefer beer i at the beginning of next month. Use matrix multiplication to relate the following:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Group B

3 Prove that matrix multiplication is associative.

4 Show that for any two matrices A and B, $(AB)^T = B^T A^T$.

5 An $n \times n$ matrix A is symmetric if $A = A^T$.

a Show that for any $n \times n$ matrix, AA^T is a symmetric matrix.

b Show that for any $n \times n$ matrix A, $(A + A^T)$ is a symmetric matrix.

6 Suppose that A and B are both $n \times n$ matrices. Show that computing the matrix product AB requires n^3 multiplications and $n^3 - n^2$ additions.

7 The **trace of a matrix** is the sum of its diagonal elements.

a For any two matrices A and B, show that trace $(A + B) =$ trace $A +$ trace B.

b For any two matrices A and B for which the products AB and BA are defined, show that trace $AB =$ trace BA.

2.2 Matrices and Systems of Linear Equations

Consider a system of linear equations given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{3}$$

In Equation (3), $x_1, x_2, \ldots, x_n$ are referred to as **variables,** or unknowns, and the a_{ij}'s and b_i's are **constants.** A set of equations such as (3) is called a linear system of m equations in n variables.

DEFINITION ■ A **solution** to a linear system of m equations in n unknowns is a set of values for the unknowns that satisfies each of the system's m equations. ■

To understand linear programming, we need to know a great deal about the properties of solutions to linear equation systems. With this in mind, we will devote much effort to studying such systems.

We denote a possible solution to Equation (3) by an n-dimensional column vector **x,** in which the ith element of **x** is the value of x_i. The following example illustrates the concept of a solution to a linear system.

EXAMPLE 5 Solution to Linear System

Show that

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is a solution to the linear system

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 - x_2 &= 0 \end{aligned} \tag{4}$$

and that

$$\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

is not a solution to linear system (4).

Solution To show that

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is a solution to Equation (4), we substitute $x_1 = 1$ and $x_2 = 2$ in both equations and check that they are satisfied: $1 + 2(2) = 5$ and $2(1) - 2 = 0$.

The vector

$$\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

is not a solution to (4), because $x_1 = 3$ and $x_2 = 1$ fail to satisfy $2x_1 - x_2 = 0$.

Using matrices can greatly simplify the statement and solution of a system of linear equations. To show how matrices can be used to compactly represent Equation (3), let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Then (3) may be written as

$$A\mathbf{x} = \mathbf{b} \tag{5}$$

Observe that both sides of Equation (5) will be $m \times 1$ matrices (or $m \times 1$ column vectors). For the matrix $A\mathbf{x}$ to equal the matrix $\mathbf{b}$ (or for the vector $A\mathbf{x}$ to equal the vector $\mathbf{b}$), their corresponding elements must be equal. The first element of $A\mathbf{x}$ is the scalar product of row 1 of A with $\mathbf{x}$. This may be written as

$$[a_{11} \quad a_{12} \quad \cdots \quad a_{1n}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

This must equal the first element of $\mathbf{b}$ (which is b_1). Thus, (5) implies that $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$. This is the first equation of (3). Similarly, (5) implies that the scalar product of row i of A with $\mathbf{x}$ must equal b_i, and this is just the ith equation of (3). Our discussion shows that (3) and (5) are two different ways of writing the same linear system. We call (5) the **matrix representation** of (3). For example, the matrix representation of (4) is

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Sometimes we abbreviate (5) by writing

$$A|\mathbf{b} \tag{6}$$

If A is an $m \times n$ matrix, it is assumed that the variables in (6) are $x_1, x_2, \ldots, x_n$. Then (6) is still another representation of (3). For instance, the matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

represents the system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 2 \\ x_2 + 2x_3 &= 3 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

PROBLEM

Group A

1 Use matrices to represent the following system of equations in two different ways:

$$\begin{aligned} x_1 - x_2 &= 4 \\ 2x_1 + x_2 &= 6 \\ x_1 + 3x_2 &= 8 \end{aligned}$$

2.3 The Gauss–Jordan Method for Solving Systems of Linear Equations

We develop in this section an efficient method (the Gauss–Jordan method) for solving a system of linear equations. Using the Gauss–Jordan method, we show that any system of linear equations must satisfy one of the following three cases:

Case 1 The system has no solution.

Case 2 The system has a unique solution.

Case 3 The system has an infinite number of solutions.

The Gauss–Jordan method is also important because many of the manipulations used in this method are used when solving linear programming problems by the simplex algorithm (see Chapter 4).

Elementary Row Operations

Before studying the Gauss–Jordan method, we need to define the concept of an **elementary row operation** (ERO). An ERO transforms a given matrix A into a new matrix A' via one of the following operations.

Type 1 ERO

A' is obtained by multiplying any row of A by a nonzero scalar. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

then a Type 1 ERO that multiplies row 2 of A by 3 would yield

$$A' = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 9 & 15 & 18 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Type 2 ERO

Begin by multiplying any row of A (say, row i) by a nonzero scalar c. For some $j \neq i$, let row j of $A' = c$(row i of A) + row j of A, and let the other rows of A' be the same as the rows of A.

For example, we might multiply row 2 of A by 4 and replace row 3 of A by 4(row 2 of A) + row 3 of A. Then row 3 of A' becomes

$$4\,[1 \quad 3 \quad 5 \quad 6] + [0 \quad 1 \quad 2 \quad 3] = [4 \quad 13 \quad 22 \quad 27]$$

and

$$A' = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \\ 4 & 13 & 22 & 27 \end{bmatrix}$$

Type 3 ERO

Interchange any two rows of A. For instance, if we interchange rows 1 and 3 of A, we obtain

$$A' = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 5 & 6 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Type 1 and Type 2 EROs formalize the operations used to solve a linear equation system. To solve the system of equations

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 + 4x_2 &= 7 \end{aligned} \tag{7}$$

we might proceed as follows. First replace the second equation in (7) by -2(first equation in (7)) + second equation in (7). This yields the following linear system:

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_2 &= 3 \end{aligned} \tag{7.1}$$

Then multiply the second equation in (7.1) by $\frac{1}{2}$, yielding the system

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_2 &= \tfrac{3}{2} \end{aligned} \tag{7.2}$$

Finally, replace the first equation in (7.2) by -1[second equation in (7.2)] + first equation in (7.2). This yields the system

$$\begin{aligned} x_1 \quad &= \tfrac{1}{2} \\ x_2 &= \tfrac{3}{2} \end{aligned} \tag{7.3}$$

System (7.3) has the unique solution $x_1 = \frac{1}{2}$ and $x_2 = \frac{3}{2}$. The systems (7), (7.1), (7.2), and (7.3) are *equivalent* in that they have the same set of solutions. This means that $x_1 = \frac{1}{2}$ and $x_2 = \frac{3}{2}$ is also the unique solution to the original system, (7).

If we view (7) in the augmented matrix form $(A|\mathbf{b})$, we see that the steps used to solve (7) may be seen as Type 1 and Type 2 EROs applied to $A|\mathbf{b}$. Begin with the augmented matrix version of (7):

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 4 & 7 \end{array}\right] \tag{7$'$}$$

Now perform a Type 2 ERO by replacing row 2 of (7′) by -2(row 1 of (7′)) + row 2 of (7′). The result is

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 2 & 3 \end{array}\right] \tag{7.1$'$}$$

which corresponds to (7.1). Next, we multiply row 2 of (7.1′) by $\frac{1}{2}$ (a Type 1 ERO), resulting in

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} \end{array}\right] \tag{7.2$'$}$$

which corresponds to (7.2). Finally, perform a Type 2 ERO by replacing row 1 of (7.2′) by -1(row 2 of (7.2′)) + row 1 of (7.2′). The result is

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \end{array}\right] \tag{7.3$'$}$$

which corresponds to (7.3). Translating (7.3′) back into a linear system, we obtain the system $x_1 = \frac{1}{2}$ and $x_2 = \frac{3}{2}$, which is identical to (7.3).

Finding a Solution by the Gauss–Jordan Method

The discussion in the previous section indicates that if the matrix $A'|\mathbf{b}'$ is obtained from $A|\mathbf{b}$ via an ERO, the systems $A\mathbf{x} = \mathbf{b}$ and $A'\mathbf{x} = \mathbf{b}'$ are equivalent. Thus, any sequence of EROs performed on the augmented matrix $A|\mathbf{b}$ corresponding to the system $A\mathbf{x} = \mathbf{b}$ will yield an equivalent linear system.

The Gauss–Jordan method solves a linear equation system by utilizing EROs in a systematic fashion. We illustrate the method by finding the solution to the following linear system:

$$\begin{aligned} 2x_1 + 2x_2 + x_3 &= 9 \\ 2x_1 - x_2 + 2x_3 &= 6 \\ x_1 - x_2 + 2x_3 &= 5 \end{aligned} \tag{8}$$

The augmented matrix representation is

$$A|\mathbf{b} = \left[\begin{array}{ccc|c} 2 & 2 & 1 & 9 \\ 2 & -1 & 2 & 6 \\ 1 & -1 & 2 & 5 \end{array}\right] \tag{8$'$}$$

Suppose that by performing a sequence of EROs on (8′) we could transform (8′) into

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \tag{9'}$$

We note that the result obtained by performing an ERO on a system of equations can also be obtained by multiplying both sides of the matrix representation of the system of equations by a particular matrix. This explains why EROs do not change the set of solutions to a system of equations.

Matrix (9′) corresponds to the following linear system:

$$\begin{aligned} x_1 \qquad &= 1 \\ x_2 \quad &= 2 \\ x_3 &= 3 \end{aligned} \tag{9}$$

System (9) has the unique solution $x_1 = 1$, $x_2 = 2$, $x_3 = 3$. Because (9′) was obtained from (8′) by a sequence of EROs, we know that (8) and (9) are equivalent linear systems. Thus, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ must also be the unique solution to (8). We now show how we can use EROs to transform a relatively complicated system such as (8) into a relatively simple system like (9). This is the essence of the Gauss–Jordan method.

We begin by using EROs to transform the first column of (8′) into

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Then we use EROs to transform the second column of the resulting matrix into

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Finally, we use EROs to transform the third column of the resulting matrix into

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

As a final result, we will have obtained (9′). We now use the Gauss–Jordan method to solve (8). We begin by using a Type 1 ERO to change the element of (8′) in the first row and first column into a 1. Then we add multiples of row 1 to row 2 and then to row 3 (these are Type 2 EROs). The purpose of these Type 2 EROs is to put zeros in the rest of the first column. The following sequence of EROs will accomplish these goals.

Step 1 Multiply row 1 of (8′) by $\frac{1}{2}$. This Type 1 ERO yields

$$A_1|\mathbf{b}_1 = \left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & \frac{9}{2} \\ 2 & -1 & 2 & 6 \\ 1 & -1 & 2 & 5 \end{array}\right]$$

Step 2 Replace row 2 of $A_1|\mathbf{b}_1$ by -2(row 1 of $A_1|\mathbf{b}_1$) + row 2 of $A_1|\mathbf{b}_1$. The result of this Type 2 ERO is

$$A_2|\mathbf{b}_2 = \left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & -3 & 1 & -3 \\ 1 & -1 & 2 & 5 \end{array}\right]$$

Step 3 Replace row 3 of $A_2|\mathbf{b}_2$ by -1(row 1 of $A_2|\mathbf{b}_2$ + row 3 of $A_2|\mathbf{b}_2$. The result of this Type 2 ERO is

$$A_3|\mathbf{b}_3 = \left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & -3 & 1 & -3 \\ 0 & -2 & \frac{3}{2} & \frac{1}{2} \end{array}\right]$$

The first column of (8′) has now been transformed into

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

By our procedure, we have made sure that the variable x_1 occurs in only a single equation and in that equation has a coefficient of 1. We now transform the second column of $A_3|\mathbf{b}_3$ into

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We begin by using a Type 1 ERO to create a 1 in row 2 and column 2 of $A_3|\mathbf{b}_3$. Then we use the resulting row 2 to perform the Type 2 EROs that are needed to put zeros in the rest of column 2. Steps 4–6 accomplish these goals.

Step 4 Multiply row 2 of $A_3|\mathbf{b}_3$ by $-\frac{1}{3}$.The result of this Type 1 ERO is

$$A_4|\mathbf{b}_4 = \left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & -2 & \frac{3}{2} & \frac{1}{2} \end{array}\right]$$

Step 5 Replace row 1 of $A_4|\mathbf{b}_4$ by -1(row 2 of $A_4|\mathbf{b}_4$) + row 1 of $A_4|\mathbf{b}_4$. The result of this Type 2 ERO is

$$A_5|\mathbf{b}_5 = \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{6} & \frac{7}{2} \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & -2 & \frac{3}{2} & \frac{1}{2} \end{array}\right]$$

Step 6 Replace row 3 of $A_5|\mathbf{b}_5$ by 2(row 2 of $A_5|\mathbf{b}_5$) + row 3 of $A_5|\mathbf{b}_5$. The result of this Type 2 ERO is

$$A_6|\mathbf{b}_6 = \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{6} & \frac{7}{2} \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & 0 & \frac{5}{6} & \frac{5}{2} \end{array}\right]$$

Column 2 has now been transformed into

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Observe that our transformation of column 2 did not change column 1.

To complete the Gauss–Jordan procedure, we must transform the third column of $A_6|\mathbf{b}_6$ into

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We first use a Type 1 ERO to create a 1 in the third row and third column of $A_6|\mathbf{b}_6$. Then we use Type 2 EROs to put zeros in the rest of column 3. Steps 7–9 accomplish these goals.

Step 7 Multiply row 3 of $A_6|\mathbf{b}_6$ by $\frac{6}{5}$. The result of this Type 1 ERO is

$$A_7|\mathbf{b}_7 = \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{6} & \frac{7}{2} \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

Step 8 Replace row 1 of $A_7|\mathbf{b}_7$ by $-\frac{5}{6}$(row 3 of $A_7|\mathbf{b}_7$) + row 1 of $A_7|\mathbf{b}_7$. The result of this Type 2 ERO is

$$A_8|\mathbf{b}_8 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Step 9 Replace row 2 of $A_8|\mathbf{b}_8$ by $\frac{1}{3}$(row 3 of $A_8|\mathbf{b}_8$) + row 2 of $A_8|\mathbf{b}_8$. The result of this Type 2 ERO is

$$A_9|\mathbf{b}_9 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$A_9|\mathbf{b}_9$ represents the system of equations

$$\begin{aligned} x_1 \qquad\quad &= 1 \\ x_2 \qquad &= 2 \\ x_3 &= 3 \end{aligned} \tag{9}$$

Thus, (9) has the unique solution $x_1 = 1, x_2 = 2, x_3 = 3$. Because (9) was obtained from (8) via EROs, the unique solution to (8) must also be $x_1 = 1, x_2 = 2, x_3 = 3$.

The reader might be wondering why we defined Type 3 EROs (interchanging of rows). To see why a Type 3 ERO might be useful, suppose you want to solve

$$\begin{aligned} 2x_2 + x_3 &= 6 \\ x_1 + x_2 - x_3 &= 2 \\ 2x_1 + x_2 + x_3 &= 4 \end{aligned} \tag{10}$$

To solve (10) by the Gauss–Jordan method, first form the augmented matrix

$$A|\mathbf{b} = \left[\begin{array}{ccc|c} 0 & 2 & 1 & 6 \\ 1 & 1 & -1 & 2 \\ 2 & 1 & 1 & 4 \end{array}\right]$$

The 0 in row 1 and column 1 means that a Type 1 ERO cannot be used to create a 1 in row 1 and column 1. If, however, we interchange rows 1 and 2 (a Type 3 ERO), we obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 2 & 1 & 6 \\ 2 & 1 & 1 & 4 \end{array}\right] \tag{10$'$}$$

Now we may proceed as usual with the Gauss–Jordan method.

Special Cases: No Solution or an Infinite Number of Solutions

Some linear systems have no solution, and some have an infinite number of solutions. The following two examples illustrate how the Gauss–Jordan method can be used to recognize these cases.

EXAMPLE 6 Linear System with No Solution

Find all solutions to the following linear system:

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 2x_1 + 4x_2 &= 4 \end{aligned} \tag{11}$$

Solution We apply the Gauss–Jordan method to the matrix

$$A|\mathbf{b} = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 4 \end{array}\right]$$

We begin by replacing row 2 of $A|\mathbf{b}$ by -2(row 1 of $A|\mathbf{b}$) + row 2 of $A|\mathbf{b}$. The result of this Type 2 ERO is

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & -2 \end{array}\right] \tag{12}$$

We would now like to transform the second column of (12) into

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

but this is not possible. System (12) is equivalent to the following system of equations:

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 0x_1 + 0x_2 &= -2 \end{aligned} \tag{12$'$}$$

Whatever values we give to x_1 and x_2, the second equation in (12′) can never be satisfied. Thus, (12′) has no solution. Because (12′) was obtained from (11) by use of EROs, (11) also has no solution.

Example 6 illustrates the following idea: *If you apply the Gauss–Jordan method to a linear system and obtain a row of the form* $[0 \quad 0 \quad \cdots \quad 0|c]$ $(c \neq 0)$, *then the original linear system has no solution.*

EXAMPLE 7 Linear System with Infinite Number of Solutions

Apply the Gauss–Jordan method to the following linear system:

$$\begin{aligned} x_1 + x_2 \phantom{{}+x_3} &= 1 \\ x_2 + x_3 &= 3 \\ x_1 + 2x_2 + x_3 &= 4 \end{aligned} \tag{13}$$

Solution The augmented matrix form of (13) is

$$A|\mathbf{b} = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 2 & 1 & 4 \end{array}\right]$$

We begin by replacing row 3 (because the row 2, column 1 value is already 0) of $A|\mathbf{b}$ by -1(row 1 of $A|\mathbf{b}$) + row 3 of $A|\mathbf{b}$. The result of this Type 2 ERO is

$$A_1|\mathbf{b}_1 = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{array}\right] \tag{14}$$

Next we replace row 1 of $A_1|\mathbf{b}_1$ by -1(row 2 of $A_1|\mathbf{b}_1$) + row 1 of $A_1|\mathbf{b}_1$. The result of this Type 2 ERO is

$$A_2|\mathbf{b}_2 = \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

Now we replace row 3 of $A_2|\mathbf{b}_2$ by -1(row 2 of $A_2|\mathbf{b}_2$) + row 3 of $A_2|\mathbf{b}_2$. The result of this Type 2 ERO is

$$A_3|\mathbf{b}_3 = \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We would now like to transform the third column of $A_3|\mathbf{b}_3$ into

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

but this is not possible. The linear system corresponding to $A_3|\mathbf{b}_3$ is

$$x_1 \qquad - x_3 = -2 \tag{14.1}$$

$$x_2 + x_3 = 3 \tag{14.2}$$

$$0x_1 + 0x_2 + 0x_3 = 0 \tag{14.3}$$

Suppose we assign an arbitrary value k to x_3. Then (14.1) will be satisfied if $x_1 - k = -2$, or $x_1 = k - 2$. Similarly, (14.2) will be satisfied if $x_2 + k = 3$, or $x_2 = 3 - k$. Of course, (14.3) will be satisfied for any values of x_1, x_2, and x_3. Thus, for any number k, $x_1 = k - 2$, $x_2 = 3 - k$, $x_3 = k$ is a solution to (14). Thus, (14) has an infinite number of solutions (one for each number k). Because (14) was obtained from (13) via EROs, (13) also has an infinite number of solutions. A more formal characterization of linear systems that have an infinite number of solutions will be given after the following summary of the Gauss–Jordan method.

Summary of the Gauss–Jordan Method

Step 1 To solve $A\mathbf{x} = \mathbf{b}$, write down the augmented matrix $A|\mathbf{b}$.

Step 2 At any stage, define a current row, current column, and current entry (the entry in the current row and column). Begin with row 1 as the current row, column 1 as the current column, and a_{11} as the current entry. (**a**) If a_{11} (the current entry) is nonzero, then use EROs to transform column 1 (the current column) to

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Then obtain the new current row, column, and entry by moving down one row and one column to the right, and go to step 3. (**b**) If a_{11} (the current entry) equals 0, then do a Type 3 ERO involving the current row and any row that contains a nonzero number in the current column. Use EROs to transform column 1 to

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Then obtain the new current row, column, and entry by moving down one row and one column to the right. Go to step 3. (**c**) If there are no nonzero numbers in the first column, then obtain a new current column and entry by moving one column to the right. Then go to step 3.

Step 3 (**a**) If the new current entry is nonzero, then use EROs to transform it to 1 and the rest of the current column's entries to 0. When finished, obtain the new current row, column, and entry. If this is impossible, then stop. Otherwise, repeat step 3. (**b**) If the current entry is 0, then do a Type 3 ERO with the current row and any row that contains a nonzero number in the current column. Then use EROs to transform that current entry to 1 and the rest of the current column's entries to 0. When finished, obtain the new current row, column, and entry. If this is impossible, then stop. Otherwise, repeat step 3. (**c**) If the current column has no nonzero numbers below the current row, then obtain the new current column and entry, and repeat step 3. If it is impossible, then stop.

This procedure may require "passing over" one or more columns without transforming them (see Problem 8).

Step 4 Write down the system of equations $A'\mathbf{x} = \mathbf{b}'$ that corresponds to the matrix $A'|\mathbf{b}'$ obtained when step 3 is completed. Then $A'\mathbf{x} = \mathbf{b}'$ will have the same set of solutions as $A\mathbf{x} = \mathbf{b}$.

Basic Variables and Solutions to Linear Equation Systems

To describe the set of solutions to $A'\mathbf{x} = \mathbf{b}'$ (and $A\mathbf{x} = \mathbf{b}$), we need to define the concepts of basic and nonbasic variables.

DEFINITION ■ After the Gauss–Jordan method has been applied to any linear system, a variable that appears with a coefficient of 1 in a single equation and a coefficient of 0 in all other equations is called a **basic variable** (BV). ■

Any variable that is not a basic variable is called a **nonbasic variable** (NBV). ■

Let *BV* be the set of basic variables for $A'\mathbf{x} = \mathbf{b}'$ and NBV be the set of nonbasic variables for $A'\mathbf{x} = \mathbf{b}'$. The character of the solutions to $A'\mathbf{x} = \mathbf{b}'$ depends on which of the following cases occurs.

Case 1 $A'\mathbf{x} = \mathbf{b}'$ has at least one row of form $[0 \quad 0 \quad \cdots \quad 0|c]$ $(c \neq 0)$. Then $A\mathbf{x} = \mathbf{b}$ has no solution (recall Example 6). As an example of Case 1, suppose that when the Gauss–Jordan method is applied to the system $A\mathbf{x} = \mathbf{b}$, the following matrix is obtained:

$$
A'|\mathbf{b}' = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]
$$

In this case, $A'\mathbf{x} = \mathbf{b}'$ (and $A\mathbf{x} = \mathbf{b}$) has no solution.

Case 2 Suppose that Case 1 does not apply and NBV, the set of nonbasic variables, is empty. Then $A'\mathbf{x} = \mathbf{b}'$ (and $A\mathbf{x} = \mathbf{b}$) will have a unique solution. To illustrate this, we recall that in solving

$$
\begin{aligned}
2x_1 + 2x_2 + x_3 &= 9 \\
2x_1 - x_2 + 2x_3 &= 6 \\
x_1 - x_2 + 2x_3 &= 5
\end{aligned}
$$

the Gauss–Jordan method yielded

$$
A'|\mathbf{b}' = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]
$$

In this case, BV $= \{x_1, x_2, x_3\}$ and NBV is empty. Then the unique solution to $A'\mathbf{x} = \mathbf{b}'$ (and $A\mathbf{x} = \mathbf{b}$) is $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

Case 3 Suppose that Case 1 does not apply and NBV is nonempty. Then $A'\mathbf{x} = \mathbf{b}'$ (and $A\mathbf{x} = \mathbf{b}$) will have an infinite number of solutions. To obtain these, first assign each nonbasic variable an arbitrary value. Then solve for the value of each basic variable in terms of the nonbasic variables. For example, suppose

$$
A'|\mathbf{b}' = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{15}
$$

Because Case 1 does not apply, and BV $= \{x_1, x_2, x_3\}$ and NBV $= \{x_4, x_5\}$, we have an example of Case 3: $A'\mathbf{x} = \mathbf{b}'$ (and $A\mathbf{x} = \mathbf{b}$) will have an infinite number of solutions. To see what these solutions look like, write down $A'\mathbf{x} = \mathbf{b}'$:

$$
\begin{aligned}
x_1 \qquad\qquad\qquad + x_4 + x_5 &= 3 && (15.1) \\
x_2 \qquad + 2x_4 \qquad &= 2 && (15.2) \\
x_3 \qquad + x_5 &= 1 && (15.3) \\
0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 &= 0 && (15.4)
\end{aligned}
$$

Now assign the nonbasic variables (x_4 and x_5) arbitrary values c and k, with $x_4 = c$ and $x_5 = k$. From (15.1), we find that $x_1 = 3 - c - k$. From (15.2), we find that $x_2 = 2 - 2c$. From (15.3), we find that $x_3 = 1 - k$. Because (15.4) holds for all values of the variables, $x_1 = 3 - c - k$, $x_2 = 2 - 2c$, $x_3 = 1 - k$, $x_4 = c$, and $x_5 = k$ will, for any values of c and k, be a solution to $A'\mathbf{x} = \mathbf{b}'$ (and $A\mathbf{x} = \mathbf{b}$).

Our discussion of the Gauss–Jordan method is summarized in Figure 6. We have devoted so much time to the Gauss–Jordan method because, in our study of linear programming, examples of Case 3 (linear systems with an infinite number of solutions) will occur repeatedly. Because the end result of the Gauss–Jordan method must always be one of Cases 1–3, we have shown that any linear system will have no solution, a unique solution, or an infinite number of solutions.

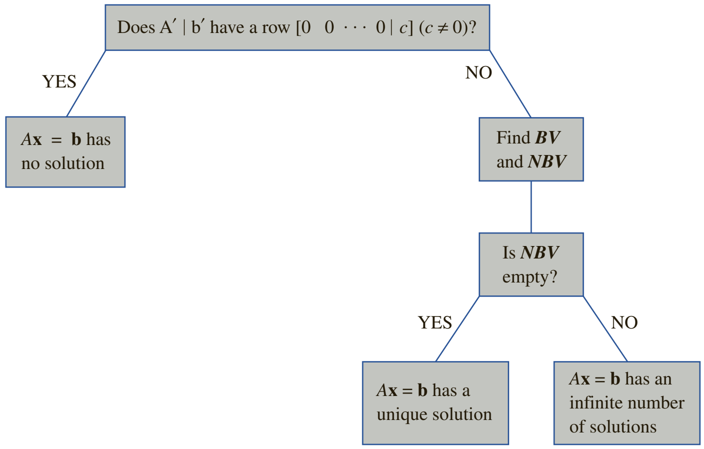

FIGURE 6
Description of Gauss–Jordan Method for Solving Linear Equations

PROBLEMS

Group A

Use the Gauss–Jordan method to determine whether each of the following linear systems has no solution, a unique solution, or an infinite number of solutions. Indicate the solutions (if any exist).

1 $$\begin{aligned} x_1 + x_2 + x_4 &= 3 \\ x_2 + x_3 &= 4 \\ x_1 + 2x_2 + x_3 + x_4 &= 8 \end{aligned}$$

2 $$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ x_1 + 2x_2 &= 6 \end{aligned}$$

3 $$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + x_2 &= 3 \\ 3x_1 + 2x_2 &= 4 \end{aligned}$$

4 $$\begin{aligned} 2x_1 - x_2 + x_3 + x_4 &= 6 \\ x_1 + x_2 + x_3 &= 4 \end{aligned}$$

5 $$\begin{aligned} x_1 + x_4 &= 5 \\ x_2 + 2x_4 &= 5 \\ x_3 + 0.5x_4 &= 1 \\ 2x_3 + x_4 &= 3 \end{aligned}$$

6 $$\begin{aligned} 2x_2 + 2x_3 &= 4 \\ x_1 + 2x_2 + x_3 &= 4 \\ x_2 - x_3 &= 0 \end{aligned}$$

7 $$\begin{aligned} x_1 + x_2 &= 2 \\ -x_2 + 2x_3 &= 3 \\ x_2 + x_3 &= 3 \end{aligned}$$

8 $$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_2 + 2x_3 + x_4 &= 2 \\ x_4 &= 3 \end{aligned}$$

Group B

9 Suppose that a linear system $A\mathbf{x} = \mathbf{b}$ has more variables than equations. Show that $A\mathbf{x} = \mathbf{b}$ cannot have a unique solution.

2.4 Linear Independence and Linear Dependence†

In this section, we discuss the concepts of a linearly independent set of vectors, a linearly dependent set of vectors, and the rank of a matrix. These concepts will be useful in our study of matrix inverses.

Before defining a linearly independent set of vectors, we need to define a linear combination of a set of vectors. Let $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of row vectors all of which have the same dimension.

†This section covers topics that may be omitted with no loss of continuity.

DEFINITION ■ A **linear combination** of the vectors in V is any vector of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \cdots + c_k\mathbf{v}_k$, where $c_1, c_2, \ldots, c_k$ are arbitrary scalars. ■

For example, if $V = \{[1 \quad 2], [2 \quad 1]\}$, then

$$\begin{aligned} 2\mathbf{v}_1 - \mathbf{v}_2 &= 2([1 \quad 2]) - [2 \quad 1] = [0 \quad 3] \\ \mathbf{v}_1 + 3\mathbf{v}_2 &= [1 \quad 2] + 3([2 \quad 1]) = [7 \quad 5] \\ 0\mathbf{v}_1 + 3\mathbf{v}_2 &= [0 \quad 0] + 3([2 \quad 1]) = [6 \quad 3] \end{aligned}$$

are linear combinations of vectors in V. The foregoing definition may also be applied to a set of column vectors.

Suppose we are given a set $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of m-dimensional row vectors. Let $\mathbf{0} = [0 \quad 0 \quad \cdots \quad 0]$ be the m-dimensional $\mathbf{0}$ vector. To determine whether V is a linearly independent set of vectors, we try to find a linear combination of the vectors in V that adds up to $\mathbf{0}$. Clearly, $0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_k$ is a linear combination of vectors in V that adds up to $\mathbf{0}$. We call the linear combination of vectors in V for which $c_1 = c_2 = \cdots = c_k = 0$ the *trivial* linear combination of vectors in V. We may now define linearly independent and linearly dependent sets of vectors.

DEFINITION ■ A set V of m-dimensional vectors is **linearly independent** if the only linear combination of vectors in V that equals $\mathbf{0}$ is the trivial linear combination. ■

A set V of m-dimensional vectors is **linearly dependent** if there is a nontrivial linear combination of the vectors in V that adds up to $\mathbf{0}$. ■

The following examples should clarify these definitions.

EXAMPLE 8 0 Vector Makes Set LD

Show that any set of vectors containing the $\mathbf{0}$ vector is a linearly dependent set.

Solution To illustrate, we show that if $V = \{[0 \quad 0], [1 \quad 0], [0 \quad 1]\}$, then V is linearly dependent, because if, say, $c_1 \neq 0$, then $c_1([0 \quad 0]) + 0([1 \quad 0]) + 0([0 \quad 1]) = [0 \quad 0]$. Thus, there is a nontrivial linear combination of vectors in V that adds up to $\mathbf{0}$.

EXAMPLE 9 LI Set of Vectors

Show that the set of vectors $V = \{[1 \quad 0], [0 \quad 1]\}$ is a linearly independent set of vectors.

Solution We try to find a nontrivial linear combination of the vectors in V that yields $\mathbf{0}$. This requires that we find scalars c_1 and c_2 (at least one of which is nonzero) satisfying $c_1([1 \quad 0]) + c_2([0 \quad 1]) = [0 \quad 0]$. Thus, c_1 and c_2 must satisfy $[c_1 \quad c_2] = [0 \quad 0]$. This implies $c_1 = c_2 = 0$. The only linear combination of vectors in V that yields $\mathbf{0}$ is the trivial linear combination. Therefore, V is a linearly independent set of vectors.

EXAMPLE 10 LD Set of Vectors

Show that $V = \{[1 \quad 2], [2 \quad 4]\}$ is a linearly dependent set of vectors.

Solution Because $2([1 \quad 2]) - 1([2 \quad 4]) = [0 \quad 0]$, there is a nontrivial linear combination with $c_1 = 2$ and $c_2 = -1$ that yields $\mathbf{0}$. Thus, V is a linearly dependent set of vectors.

Intuitively, what does it mean for a set of vectors to be linearly dependent? To understand the concept of linear dependence, observe that a set of vectors V is linearly dependent (as

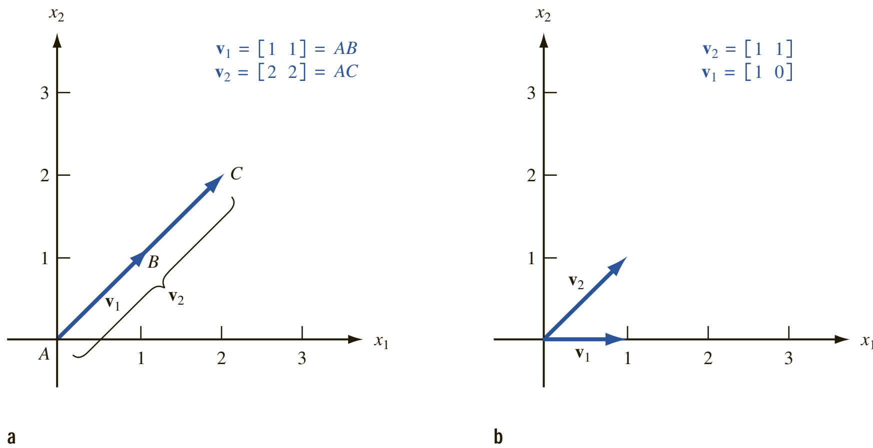

FIGURE 7
(a) Two Linearly Dependent Vectors
(b) Two Linearly Independent Vectors

long as $\mathbf{0}$ is not in V) if and only if some vector in V can be written as a nontrivial linear combination of other vectors in V (see Problem 9 at the end of this section). For instance, in Example 10, $[2 \quad 4] = 2([1 \quad 2])$. Thus, if a set of vectors V is linearly dependent, the vectors in V are, in some way, not all "different" vectors. By "different" we mean that the direction specified by any vector in V cannot be expressed by adding together multiples of other vectors in V. For example, in two dimensions it can be shown that two vectors are linearly dependent if and only if they lie on the same line (see Figure 7).

The Rank of a Matrix

The Gauss–Jordan method can be used to determine whether a set of vectors is linearly independent or linearly dependent. Before describing how this is done, we define the concept of the rank of a matrix.

Let A be any $m \times n$ matrix, and denote the rows of A by $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$. Also define $R = \{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\}$.

DEFINITION ■ The **rank** of A is the number of vectors in the largest linearly independent subset of R. ■

The following three examples illustrate the concept of rank.

EXAMPLE 11 Matrix with 0 Rank

Show that rank $A = 0$ for the following matrix:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution For the set of vectors $R = \{[0 \quad 0], [0, \quad 0]\}$, it is impossible to choose a subset of R that is linearly independent (recall Example 8).

EXAMPLE 12 Matrix with Rank of 1

Show that rank $A = 1$ for the following matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Solution Here $R = \{[1 \quad 1], [2 \quad 2]\}$. The set $\{[1 \quad 1]\}$ is a linearly independent subset of R, so rank A must be at least 1. If we try to find two linearly independent vectors in R, we fail because $2([1 \quad 1]) - [2 \quad 2] = [0 \quad 0]$. This means that rank A cannot be 2. Thus, rank A must equal 1.

EXAMPLE 13 **Matrix with Rank of 2**

Show that rank $A = 2$ for the following matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution Here $R = \{[1 \quad 0], [0 \quad 1]\}$. From Example 9, we know that R is a linearly independent set of vectors. Thus, rank $A = 2$.

To find the rank of a given matrix A, simply apply the Gauss–Jordan method to the matrix A. Let the final result be the matrix $\bar{A}$. It can be shown that performing a sequence of EROs on a matrix does not change the rank of the matrix. This implies that rank $A =$ rank $\bar{A}$. It is also apparent that the rank of $\bar{A}$ will be the number of nonzero rows in $\bar{A}$. Combining these facts, we find that rank $A =$ rank $\bar{A} =$ number of nonzero rows in $\bar{A}$.

EXAMPLE 14 **Using Gauss–Jordan Method to Find Rank of Matrix**

Find

$$\text{rank } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

Solution The Gauss–Jordan method yields the following sequence of matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \bar{A}$$

Thus, rank $A =$ rank $\bar{A} = 3$.

How to Tell Whether a Set of Vectors Is Linearly Independent

We now describe a method for determining whether a set of vectors $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is linearly independent.

Form the matrix A whose ith row is $\mathbf{v}_i$. A will have m rows. If rank $A = m$, then V is a linearly independent set of vectors, whereas if rank $A < m$, then V is a linearly dependent set of vectors.

EXAMPLE 15 **A Linearly Dependent Set of Vectors**

Determine whether $V = \{[1 \quad 0 \quad 0], [0 \quad 1 \quad 0], [1 \quad 1 \quad 0]\}$ is a linearly independent set of vectors.

Solution The Gauss–Jordan method yields the following sequence of matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \bar{A}$$

Thus, rank A = rank $\bar{A} = 2 < 3$. This shows that V is a linearly dependent set of vectors. In fact, the EROs used to transform A to $\bar{A}$ can be used to show that $[1 \quad 1 \quad 0] = [1 \quad 0 \quad 0] + [0 \quad 1 \quad 0]$. This equation also shows that V is a linearly dependent set of vectors.

PROBLEMS

Group A

Determine if each of the following sets of vectors is linearly independent or linearly dependent.

1 $V = \{[1 \quad 0 \quad 1], [1 \quad 2 \quad 1], [2 \quad 2 \quad 2]\}$

2 $V = \{[2 \quad 1 \quad 0], [1 \quad 2 \quad 0], [3 \quad 3 \quad 1]\}$

3 $V = \{[2 \quad 1], [1 \quad 2]\}$

4 $V = \{[2 \quad 0], [3 \quad 0]\}$

5 $V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \right\}$

6 $V = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Group B

7 Show that the linear system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ can be written as a linear combination of the columns of A.

8 Suppose there is a collection of three or more two-dimensional vectors. Provide an argument showing that the collection must be linearly dependent.

9 Show that a set of vectors V (not containing the $\mathbf{0}$ vector) is linearly dependent if and only if there exists some vector in V that can be written as a nontrivial linear combination of other vectors in V.

2.5 The Inverse of a Matrix

To solve a single linear equation such as $4x = 3$, we simply multiply both sides of the equation by the multiplicative inverse of 4, which is 4^{-1}, or $\frac{1}{4}$. This yields $4^{-1}(4x) = (4^{-1})3$, or $x = \frac{3}{4}$. (Of course, this method fails to work for the equation $0x = 3$, because zero has no multiplicative inverse.) In this section, we develop a generalization of this technique that can be used to solve "square" (number of equations = number of unknowns) linear systems. We begin with some preliminary definitions.

DEFINITION ■ A **square matrix** is any matrix that has an equal number of rows and columns. ■

The **diagonal elements** of a square matrix are those elements a_{ij} such that $i = j$. ■

A square matrix for which all diagonal elements are equal to 1 and all nondiagonal elements are equal to 0 is called an **identity matrix.** ■

The $m \times m$ identity matrix will be written as I_m. Thus,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \cdots$$

If the multiplications $I_m A$ and AI_m are defined, it is easy to show that $I_m A = AI_m = A$. Thus, just as the number 1 serves as the unit element for multiplication of real numbers, I_m serves as the unit element for multiplication of matrices.

Recall that $\frac{1}{4}$ is the multiplicative inverse of 4. This is because $4(\frac{1}{4}) = (\frac{1}{4})4 = 1$. This motivates the following definition of the inverse of a matrix.

DEFINITION ■ For a given $m \times m$ matrix A, the $m \times m$ matrix B is the **inverse** of A if

$$BA = AB = I_m \tag{16}$$

(It can be shown that if $BA = I_m$ or $AB = I_m$, then the other quantity will also equal I_m.) ■

Some square matrices do not have inverses. If there does exist an $m \times m$ matrix B that satisfies Equation (16), then we write $B = A^{-1}$. For example, if

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

the reader can verify that

$$\begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -7 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -7 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -7 \\ 1 & 0 & 2 \end{bmatrix}$$

To see why we are interested in the concept of a matrix inverse, suppose we want to solve a linear system $A\mathbf{x} = \mathbf{b}$ that has m equations and m unknowns. Suppose that A^{-1} exists. Multiplying both sides of $A\mathbf{x} = \mathbf{b}$ by A^{-1}, we see that any solution of $A\mathbf{x} = \mathbf{b}$ must also satisfy $A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$. Using the associative law and the definition of a matrix inverse, we obtain

$$(A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b}$$

$$\text{or} \qquad I_m\mathbf{x} = A^{-1}\mathbf{b}$$

$$\text{or} \qquad \mathbf{x} = A^{-1}\mathbf{b}$$

This shows that knowing A^{-1} enables us to find the unique solution to a square linear system. This is the analog of solving $4x = 3$ by multiplying both sides of the equation by 4^{-1}.

The Gauss–Jordan method may be used to find A^{-1} (or to show that A^{-1} does not exist). To illustrate how we can use the Gauss–Jordan method to invert a matrix, suppose we want to find A^{-1} for

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

This requires that we find a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A^{-1}$$

that satisfies

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{17}$$

From Equation (17), we obtain the following pair of simultaneous equations that must be satisfied by a, b, c, and d:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \qquad \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus, to find

$$\begin{bmatrix} a \\ c \end{bmatrix}$$

(the first column of A^{-1}), we can apply the Gauss–Jordan method to the augmented matrix

$$\left[\begin{array}{cc|c} 2 & 5 & 1 \\ 1 & 3 & 0 \end{array}\right]$$

Once EROs have transformed

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

to I_2,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

will have been transformed into the first column of A^{-1}. To determine

$$\begin{bmatrix} b \\ d \end{bmatrix}$$

(the second column of A^{-1}), we apply EROs to the augmented matrix

$$\left[\begin{array}{cc|c} 2 & 5 & 0 \\ 1 & 3 & 1 \end{array}\right]$$

When

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

has been transformed into I_2,

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

will have been transformed into the second column of A^{-1}. Thus, to find each column of A^{-1}, we must perform a sequence of EROs that transform

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

into I_2. This suggests that we can find A^{-1} by applying EROs to the 2×4 matrix

$$A|I_2 = \left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

When

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

has been transformed to I_2,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

will have been transformed into the first column of A^{-1}, and

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

will have been transformed into the second column of A^{-1}. Thus, *as A is transformed into I_2, I_2 is transformed into A^{-1}*. The computations to determine A^{-1} follow.

Step 1 Multiply row 1 of $A|I_2$ by $\frac{1}{2}$. This yields

$$A'|I_2' = \left[\begin{array}{cc|cc} 1 & \frac{5}{2} & \frac{1}{2} & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

Step 2 Replace row 2 of $A'|I_2'$ by -1(row 1 of $A'|I_2'$) + row 2 of $A'|I_2'$. This yields

$$A''|I_2'' = \left[\begin{array}{cc|cc} 1 & \frac{5}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right]$$

Step 3 Multiply row 2 of $A''|I_2''$ by 2. This yields

$$A'''|I_2''' = \left[\begin{array}{cc|cc} 1 & \frac{5}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

Step 4 Replace row 1 of $A'''|I_2'''$ by $-\frac{5}{2}$(row 2 of $A'''|I_2'''$) + row 1 of $A'''|I_2'''$. This yields

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

Because A has been transformed into I_2, I_2 will have been transformed into A^{-1}. Hence,

$$A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

The reader should verify that $AA^{-1} = A^{-1}A = I_2$.

A Matrix May Not Have an Inverse

Some matrices do not have inverses. To illustrate, let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \tag{18}$$

To find A^{-1} we must solve the following pair of simultaneous equations:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{18.1}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} f \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{18.2}$$

When we try to solve (18.1) by the Gauss–Jordan method, we find that

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4 & 0 \end{array}\right]$$

is transformed into

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & -2 \end{array}\right]$$

This indicates that (18.1) has no solution, and A^{-1} cannot exist.

Observe that (18.1) fails to have a solution, because the Gauss–Jordan method transforms A into a matrix with a row of zeros on the bottom. This can only happen if rank $A < 2$. If $m \times m$ matrix A has rank $A < m$, then A^{-1} will not exist.

The Gauss–Jordan Method for Inverting an $m \times m$ Matrix A

Step 1 Write down the $m \times 2m$ matrix $A|I_m$.

Step 1 Use EROs to transform $A|I_m$ into $I_m|B$. This will be possible only if rank $A = m$. In this case, $B = A^{-1}$. If rank $A < m$, then A has no inverse.

Using Matrix Inverses to Solve Linear Systems

As previously stated, matrix inverses can be used to solve a linear system $A\mathbf{x} = \mathbf{b}$ in which the number of variables and equations are equal. Simply multiply both sides of $A\mathbf{x} = \mathbf{b}$ by A^{-1} to obtain the solution $\mathbf{x} = A^{-1}\mathbf{b}$. For example, to solve

$$\begin{aligned} 2x_1 + 5x_2 &= 7 \\ x_1 + 3x_2 &= 4 \end{aligned} \tag{19}$$

write the matrix representation of (19):

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \tag{20}$$

Let

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

We found in the previous illustration that

$$A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

	A	B	C	D	E	F	G	H
1		**Inverting**						
2		**a**						
3		**Matrix**			2	0	-1	
4				A	3	1	2	
5					-1	0	1	
6								
7					1	0	1	
8				A^{-1}	-5	1	-7	
9					1	0	2	

FIGURE 8

Multiplying both sides of (20) by A^{-1}, we obtain

$$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus, $x_1 = 1$, $x_2 = 1$ is the unique solution to system (19).

Inverting Matrices with Excel

Minverse.xls

The Excel =MINVERSE command makes it easy to invert a matrix. See Figure 8 and file Minverse.xls. Suppose we want to invert the matrix

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

Simply enter the matrix in E3:G5 and select the range (we chose E7:G9) where you want A^{-1} to be computed. In the upper left-hand corner of the range E7:G9 (cell E7), we enter the formula

$$= \text{MINVERSE(E3:G5)}$$

and select **Control Shift Enter.** This enters an array function that computes A^{-1} in the range E7:G9. You cannot edit part of an array function, so if you want to delete A^{-1}, you must delete the entire range where A^{-1} is present.

PROBLEMS

Group A

Find A^{-1} (if it exists) for the following matrices:

1 $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$

2 $\begin{bmatrix} 1 & 0 & 1 \\ 4 & 1 & -2 \\ 3 & 1 & -1 \end{bmatrix}$

3 $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

4 $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 2 & 4 & 1 \end{bmatrix}$

5 Use the answer to Problem 1 to solve the following linear system:

$$\begin{aligned} x_1 + 3x_2 &= 4 \\ 2x_1 + 5x_2 &= 7 \end{aligned}$$

6 Use the answer to Problem 2 to solve the following linear system:

$$\begin{aligned} x_1 + \qquad\quad x_3 &= 4 \\ 4x_1 + x_2 - 2x_3 &= 0 \\ 3x_1 + x_2 - x_3 &= 2 \end{aligned}$$

Group B

7 Show that a square matrix has an inverse if and only if its rows form a linearly independent set of vectors.

8 Consider a square matrix B whose inverse is given by B^{-1}.

a In terms of B^{-1}, what is the inverse of the matrix $100B$?

b Let B' be the matrix obtained from B by doubling every entry in row 1 of B. Explain how we could obtain the inverse of B' from B^{-1}.

c Let B' be the matrix obtained from B by doubling every entry in column 1 of B. Explain how we could obtain the inverse of B' from B^{-1}.

9 Suppose that A and B both have inverses. Find the inverse of the matrix AB.

10 Suppose A has an inverse. Show that $(A^T)^{-1} = (A^{-1})^T$. (*Hint:* Use the fact that $AA^{-1} = I$, and take the transpose of both sides.)

11 A square matrix A is *orthogonal* if $AA^T = I$. What properties must be possessed by the columns of an orthogonal matrix?

2.6 Determinants

Associated with any square matrix A is a number called the *determinant* of A (often abbreviated as det A or $|A|$). Knowing how to compute the determinant of a square matrix will be useful in our study of nonlinear programming.

For a 1×1 matrix $A = [a_{11}]$,

$$\det A = a_{11} \tag{21}$$

For a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{22}$$
$$\det A = a_{11}a_{22} - a_{21}a_{12}$$

For example,

$$\det \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} = 2(5) - 3(4) = -2$$

Before we learn how to compute det A for larger square matrices, we need to define the concept of the *minor* of a matrix.

DEFINITION ■ If A is an $m \times m$ matrix, then for any values of i and j, the ijth **minor** of A (written A_{ij}) is the $(m-1) \times (m-1)$ submatrix of A obtained by deleting row i and column j of A. ■

For example,

$$\text{if} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \text{then} \quad A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} \quad \text{and} \quad A_{32} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$$

Let A be any $m \times m$ matrix. We may write A as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

To compute det A, pick any value of i $(i = 1, 2, \ldots, m)$ and compute det A:

$$\det A = (-1)^{i+1}a_{i1}(\det A_{i1}) + (-1)^{i+2}a_{i2}(\det A_{i2}) + \cdots + (-1)^{i+m}a_{im}(\det A_{im}) \tag{23}$$

Formula (23) is called the expansion of det A by the cofactors of row i. The virtue of (23) is that it reduces the computation of det A for an $m \times m$ matrix to computations involving only $(m-1) \times (m-1)$ matrices. Apply (23) until det A can be expressed in terms of 2×2 matrices. Then use Equation (22) to find the determinants of the relevant 2×2 matrices.

To illustrate the use of (23), we find det A for

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

We expand det A by using row 1 cofactors. Notice that $a_{11} = 1$, $a_{12} = 2$, and $a_{13} = 3$. Also

$$A_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$$

so by (22), det $A_{11} = 5(9) - 8(6) = -3$;

$$A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$$

so by (22), det $A_{12} = 4(9) - 7(6) = -6$; and

$$A_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

so by (22), det $A_{13} = 4(8) - 7(5) = -3$. Then by (23),

$$\begin{aligned} \det A &= (-1)^{1+1}a_{11}(\det A_{11}) + (-1)^{1+2}a_{12}(\det A_{12}) + (-1)^{1+3}a_{13}(\det A_{13}) \\ &= (1)(1)(-3) + (-1)(2)(-6) + (1)(3)(-3) = -3 + 12 - 9 = 0 \end{aligned}$$

The interested reader may verify that expansion of det A by either row 2 or row 3 cofactors also yields det $A = 0$.

We close our discussion of determinants by noting that they can be used to invert square matrices and to solve linear equation systems. Because we already have learned to use the Gauss–Jordan method to invert matrices and to solve linear equation systems, we will not discuss these uses of determinants.

PROBLEMS

Group A

1 Verify that det $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 0$ by using expansions by row 2 and row 3 cofactors.

2 Find det $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

3 A matrix is said to be upper triangular if for $i > j$, $a_{ij} = 0$. Show that the determinant of any upper triangular 3×3 matrix is equal to the product of the matrix's diagonal elements. (This result is true for any upper triangular matrix.)

Group B

4 **a** Show that for any 1×1 and 3×3 matrix, det $-A = -\det A$.

b Show that for any 2×2 and 4×4 matrix, det $-A = \det A$.

c Generalize the results of parts (a) and (b).

SUMMARY

Matrices

A **matrix** is any rectangular array of numbers. For the matrix A, we let a_{ij} represent the element of A in row i and column j.

A matrix with only one row or one column may be thought of as a **vector**. Vectors appear in boldface type ($\mathbf{v}$). Given a row vector $\mathbf{u} = [u_1 \quad u_2 \quad \cdots \quad u_n]$ and a column

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

of the same dimension, the **scalar product** of $\mathbf{u}$ and $\mathbf{v}$ (written $\mathbf{u} \cdot \mathbf{v}$) is the number $u_1v_1 + u_2v_2 + \cdots + u_nv_n$.

Given two matrices A and B, the **matrix product** of A and B (written AB) is defined if and only if the number of columns in A = the number of rows in B. Suppose this is the case and A has m rows and B has n columns. Then the matrix product $C = AB$ of A and B is the $m \times n$ matrix C whose ijth element is determined as follows: The ijth element of C = the scalar product of row i of A with column j of B.

Matrices and Linear Equations

The **linear equation system**

$$\begin{array}{ccccccc} a_{11}x_1 & + a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & & \vdots & = & \vdots \\ a_{m1}x_1 & + a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m \end{array}$$

may be written as $A\mathbf{x} = \mathbf{b}$ or $A|\mathbf{b}$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The Gauss–Jordan Method

Using **elementary row operations** (EROs), we may solve any linear equation system. From a matrix A, an ERO yields a new matrix A' via one of three procedures.

Type 1 ERO

Obtain A' by multiplying any row of A by a nonzero scalar.

Type 2 ERO

Multiply any row of A (say, row i) by a nonzero scalar c. For some $j \neq i$, let row j of $A' = c(\text{row } i \text{ of } A) + \text{row } j$ of A, and let the other rows of A' be the same as the rows of A.

Type 3 ERO

Interchange any two rows of A.

The Gauss–Jordan method uses EROs to solve linear equation systems, as shown in the following steps.

Step 1 To solve $A\mathbf{x} = \mathbf{b}$, write down the augmented matrix $A|\mathbf{b}$.

Step 2 Begin with row 1 as the current row, column 1 as the current column, and a_{11} as the current entry. (**a**) If a_{11} (the current entry) is nonzero, then use EROs to transform column 1 (the current column) to

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Then obtain the new current row, column, and entry by moving down one row and one column to the right, and go to step 3. (**b**) If a_{11} (the current entry) equals 0, then do a Type 3 ERO switch with any row with a nonzero value in the same column. Use EROs to transform column 1 to

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and proceed to step 3 after moving into a new current row, column, and entry. (**c**) If there are no nonzero numbers in the first column, then proceed to a new current column and entry. Then go to step 3.

Step 3 (**a**) If the current entry is nonzero, use EROs to transform it to 1 and the rest of the current column's entries to 0. Obtain the new current row, column, and entry. If this is impossible, then stop. Otherwise, repeat step 3. (**b**) If the current entry is 0, then do a Type 3 ERO switch with any row with a nonzero value in the same column. Transform the column using EROs and move to the next current entry. If this is impossible, then stop. Otherwise, repeat step 3. (**c**) If the current column has no nonzero numbers below the current row, then obtain the new current column and entry, and repeat step 3. If it is impossible, then stop.

This procedure may require "passing over" one or more columns without transforming them.

Step 4 Write down the system of equations $A'\mathbf{x} = \mathbf{b}'$ that corresponds to the matrix $A'|\mathbf{b}'$ obtained when step 3 is completed. Then $A'\mathbf{x} = \mathbf{b}'$ will have the same set of solutions as $A\mathbf{x} = \mathbf{b}$.

To describe the set of solutions to $A'\mathbf{x} = \mathbf{b}'$ (and $A\mathbf{x} = \mathbf{b}$), we define the concepts of basic and nonbasic variables. After the Gauss–Jordan method has been applied to any linear system, a variable that appears with a coefficient of 1 in a single equation and a coefficient of 0 in all other equations is called a **basic variable.** Any variable that is not a basic variable is called a **nonbasic variable.**

Let BV be the set of basic variables for $A'\mathbf{x} = \mathbf{b}'$ and NBV be the set of nonbasic variables for $A'\mathbf{x} = \mathbf{b}'$.

Case 1 $A'\mathbf{x} = \mathbf{b}'$ contains at least one row of the form $[0 \quad 0 \quad \cdots \quad 0|c]$ $(c \neq 0)$. In this case, $A\mathbf{x} = \mathbf{b}$ has no solution.

Case 2 If Case 1 does not apply and NBV, the set of nonbasic variables, is empty, then $A\mathbf{x} = \mathbf{b}$ will have a unique solution.

Case 3 If Case 1 does not hold and NBV is nonempty, then $A\mathbf{x} = \mathbf{b}$ will have an infinite number of solutions.

Linear Independence, Linear Dependence, and the Rank of a Matrix

A set V of m-dimensional vectors is **linearly independent** if the only linear combination of vectors in V that equals $\mathbf{0}$ is the trivial linear combination. A set V of m-dimensional vectors is **linearly dependent** if there is a nontrivial linear combination of the vectors in V that adds to $\mathbf{0}$.

Let A be any $m \times n$ matrix, and denote the rows of A by $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$. Also define $R = \{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\}$. The **rank** of A is the number of vectors in the largest linearly independent subset of R. To find the rank of a given matrix A, apply the Gauss–Jordan method to the matrix A. Let the final result be the matrix $\bar{A}$. Then rank A = rank $\bar{A}$ = number of nonzero rows in $\bar{A}$.

To determine if a set of vectors $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is linearly dependent, form the matrix A whose ith row is $\mathbf{v}_i$. A will have m rows. If rank $A = m$, then V is a linearly independent set of vectors; if rank $A < m$, then V is a linearly dependent set of vectors.

Inverse of a Matrix

For a given square $(m \times m)$ matrix A, if $AB = BA = I_m$, then B is the **inverse** of A (written $B = A^{-1}$). The Gauss–Jordan method for inverting an $m \times m$ matrix A to get A^{-1} is as follows:

Step 1 Write down the $m \times 2m$ matrix $A|I_m$.

Step 2 Use EROs to transform $A|I_m$ into $I_m|B$. This will only be possible if rank $A = m$. In this case, $B = A^{-1}$. If rank $A < m$, then A has no inverse.

Determinants

Associated with any square $(m \times m)$ matrix A is a number called the **determinant** of A (written det A or $|A|$). For a 1×1 matrix, det $A = a_{11}$. For a 2×2 matrix, det $A = a_{11}a_{22} - a_{21}a_{12}$. For a general $m \times m$ matrix, we can find det A by repeated application of the following formula (valid for $i = 1, 2, \ldots, m$):

$$\det A = (-1)^{i+1}a_{i1}(\det A_{i1}) + (-1)^{i+2}a_{i2}(\det A_{i2}) + \cdots + (-1)^{i+m}a_{im}(\det A_{im})$$

Here A_{ij} is the ijth **minor** of A, which is the $(m - 1) \times (m - 1)$ matrix obtained from A after deleting the ith row and jth column of A.

REVIEW PROBLEMS

Group A

1 Find all solutions to the following linear system:

$$\begin{aligned} x_1 + x_2 \phantom{{}+x_3} &= 2 \\ x_2 + x_3 &= 3 \\ x_1 + 2x_2 + x_3 &= 5 \end{aligned}$$

2 Find the inverse of the matrix $\begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$.

3 Each year, 20% of all untenured State University faculty become tenured, 5% quit, and 75% remain untenured. Each year, 90% of all tenured S.U. faculty remain tenured and 10% quit. Let U_t be the number of untenured S.U. faculty at the beginning of year t, and T_t the tenured number.

Use matrix multiplication to relate the vector $\begin{bmatrix} U_{t+1} \\ T_{t+1} \end{bmatrix}$ to the vector $\begin{bmatrix} U_t \\ T_t \end{bmatrix}$.

4 Use the Gauss–Jordan method to determine all solutions to the following linear system:

$$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ x_1 + x_2 &= 1 \\ x_1 + 2x_2 &= 2 \end{aligned}$$

5 Find the inverse of the matrix $\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$.

6 The grades of two students during their last semester at S.U. are shown in Table 2.

Courses 1 and 2 are four-credit courses, and courses 3 and 4 are three-credit courses. Let GPA_i be the semester grade point average for student i. Use matrix multiplication to express the vector $\begin{bmatrix} \text{GPA}_1 \\ \text{GPA}_2 \end{bmatrix}$ in terms of the information given in the problem.

7 Use the Gauss–Jordan method to find all solutions to the following linear system:

$$\begin{aligned} 2x_1 + x_2 &= 3 \\ 3x_1 + x_2 &= 4 \\ x_1 - x_2 &= 0 \end{aligned}$$

8 Find the inverse of the matrix $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$.

9 Let C_t = number of children in Indiana at the beginning of year t, and A_t = number of adults in Indiana at the beginning of year t. During any given year, 5% of all children become adults, and 1% of all children die. Also, during any given year, 3% of all adults die. Use matrix multiplication to express the vector $\begin{bmatrix} C_{t+1} \\ A_{t+1} \end{bmatrix}$ in terms of $\begin{bmatrix} C_t \\ A_t \end{bmatrix}$.

TABLE 2

Student	Course 1	2	3	4
1	3.6	3.8	2.6	3.4
2	2.7	3.1	2.9	3.6

10 Use the Gauss–Jordan method to find all solutions to the following linear equation system:

$$\begin{aligned} x_1 \phantom{{}+x_2} - x_3 &= 4 \\ x_2 + x_3 &= 2 \\ x_1 + x_2 &= 5 \end{aligned}$$

11 Use the Gauss–Jordan method to find the inverse of the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

12 During any given year, 10% of all rural residents move to the city, and 20% of all city residents move to a rural area (all other people stay put!). Let R_t be the number of rural residents at the beginning of year t, and C_t be the number of city residents at the beginning of year t. Use matrix multiplication to relate the vector $\begin{bmatrix} R_{t+1} \\ C_{t+1} \end{bmatrix}$ to the vector $\begin{bmatrix} R_t \\ C_t \end{bmatrix}$.

13 Determine whether the set $V = \{[1 \quad 2 \quad 1], [2 \quad 0 \quad 0]\}$ is a linearly independent set of vectors.

14 Determine whether the set $V = \{[1 \quad 0 \quad 0], [0 \quad 1 \quad 0], [-1 \quad -1 \quad 0]\}$ is a linearly independent set of vectors.

15 Let $A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$.

a For what values of a, b, c, and d will A^{-1} exist?

b If A^{-1} exists, then find it.

16 Show that the following linear system has an infinite number of solutions:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$$

17 Before paying employee bonuses and state and federal taxes, a company earns profits of \$60,000. The company pays employees a bonus equal to 5% of after-tax profits. State tax is 5% of profits (after bonuses are paid). Finally, federal tax is 40% of profits (after bonuses and state tax are paid). Determine a linear equation system to find the amounts paid in bonuses, state tax, and federal tax.

18 Find the determinant of the matrix $A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

19 Show that any 2×2 matrix A that does not have an inverse will have $\det A = 0$.

Group B

20 Let A be an $m \times m$ matrix.

a Show that if rank $A = m$, then $A\mathbf{x} = \mathbf{0}$ has a unique solution. What is the unique solution?

b Show that if rank $A < m$, then $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions.

21 Consider the following linear system:

$$[x_1 \quad x_2 \quad \cdots \quad x_n] = [x_1 \quad x_2 \quad \cdots \quad x_n]P$$

where

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

If the sum of each row of the P matrix equals 1, then use Problem 20 to show that this linear system has an infinite number of solutions.

22[†] The national economy of Seriland manufactures three products: steel, cars, and machines. (1) To produce \$1 of steel requires 30¢ of steel, 15¢ of cars, and 40¢ of machines. (2) To produce \$1 of cars requires 45¢ of steel, 20¢ of cars, and 10¢ of machines. (3) To produce \$1 of machines requires 40¢ of steel, 10¢ of cars, and 45¢ of machines. During the coming year, Seriland wants to consume d_s dollars of steel, d_c dollars of cars, and d_m dollars of machinery.

For the coming year, let

s = dollar value of steel produced
c = dollar value of cars produced
m = dollar value of machines produced

Define A to be the 3×3 matrix whose ijth element is the dollar value of product i required to produce \$1 of product j (steel = product 1, cars = product 2, machinery = product 3).

a Determine A.

b Show that

$$\begin{bmatrix} s \\ c \\ m \end{bmatrix} = A\begin{bmatrix} s \\ c \\ m \end{bmatrix} + \begin{bmatrix} d_s \\ d_c \\ d_m \end{bmatrix} \qquad (24)$$

(*Hint:* Observe that the value of next year's steel production = (next year's consumer steel demand) + (steel needed to make next year's steel) + (steel needed to make next year's cars) + (steel needed to make next year's machines). This should give you the general idea.)

c Show that Equation (24) may be rewritten as

$$(I - A)\begin{bmatrix} s \\ c \\ m \end{bmatrix} = \begin{bmatrix} d_s \\ d_c \\ d_m \end{bmatrix}$$

d Given values for d_s, d_c, and d_m, describe how you can use $(I - A)^{-1}$ to determine if Seriland can meet next year's consumer demand.

e Suppose next year's demand for steel increases by \$1. This will increase the value of the steel, cars, and machines that must be produced next year. In terms of $(I - A)^{-1}$, determine the change in next year's production requirements.

REFERENCES

The following references contain more advanced discussions of linear algebra. To understand the theory of linear and nonlinear programming, master at least one of these books:

Dantzig, G. *Linear Programming and Extensions.* Princeton, N.J.: Princeton University Press, 1963.

Hadley, G. *Linear Algebra.* Reading, Mass.: Addison-Wesley, 1961.

Strang, G. *Linear Algebra and Its Applications,* 3d ed. Orlando, Fla.: Academic Press, 1988.

Leontief, W. *Input–Output Economics.* New York: Oxford University Press, 1966.

Teichroew, D. *An Introduction to Management Science: Deterministic Models.* New York: Wiley, 1964. A more extensive discussion of linear algebra than this chapter gives (at a comparable level of difficulty).

[†]Based on Leontief (1966). See references at end of chapter.