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SINGLE OBJECT AUCTIONS WITH INTERDEPENDENT VALUES

ABSTRACT: *This paper reviews single object auctions when bidders' values of the object are interdependent. We will see how the auction forms could be ranked in terms of expected revenue when signals that bidders have about the value of the object are affiliated. In the discussion that follows we will deal with reserve prices and entry fees. Furthermore we will examine the conditions that have to be met for English auction with asymmetric bidders to allocate the object efficiently. Finally, common value auctions will be considered when all bidders have the same value for the object.*

KEY WORDS: *Affiliation; The linkage principle; Winner's curse; Common value auctions; Ex-post equilibrium.*

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1. INTRODUCTION

In this paper we will analyse single object auctions with *interdependent values*. We recommend the reader to read our previous paper[1] that deals with private value auctions. However we will reconsider the most important points of auction theory which are essential for this paper.

In private value auctions it is assumed that each bidder assigns a value to the object being auctioned, and his value is his private information. This value could be regarded as a bidder's reserve price or the maximal amount he is willing to pay for the item. Each bidder determines a bid at the auction which depends on his value.

We will consider the four most commonly used auction types. The first two are known as *open auctions* because bidders publicly submit their bids. The most commonly used auction form is the *English auction*, in which the auctioneer starts the auction with a low price and raises that price gradually until only one bidder expresses a willingness to buy at that price. The last bidder who stays in is the winner and he gets the object and pays the price at which the previous bidder dropped out. The other form of open auction is the *Dutch auction*. The auctioneer starts the auction with a high price and lowers that price gradually. The first bidder who indicates an interest in buying the item at the price posted by the auctioneer wins the auction and pays that price. The other two auction forms are known as sealed-bid auctions, because bidders submit their bids in sealed envelopes. In a *first-price sealed-bid auction* the bidder who has submitted the highest bid is the winner and he pays his bid. In a *second-price auction* the bidder who has submitted the highest bid is the winner, but he pays the second highest bid. In this paper we will consider two additional forms of sealed-bid auction. The first is the *all-pay auction* in which the bidder with the highest bid obtains the object, but all bidders pay their bids. The second is the *war of attrition* in which the bidder with the highest bid gets the object but he pays the second highest bid, whereas each losing bidder pays his bid.

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1 Trifunović (2010).

Bidders can have *private, interdependent*, *or common values* for the object. In the case of private values a value that a particular bidder assigns to the object is independent of the values of the other bidders. If a value that a particular bidder assigns to the object depends on the other bidders' values we have the *interdependent* value case. A special case of interdependent values is called the *common value* model, in which bidders have the same value for the object.

We can distinguish between *single object* auctions in which there is a single object for sale and *multiple object* auctions when there are several objects for sale.

Finally, there are two sometimes conflicting criteria that an auction mechanism has to achieve. The first is *efficiency*, which means that the object has to be sold to the bidder who has the highest value. The second is *maximization of* the *seller's expected revenue* from the sale.

This paper deals with single object auctions with interdependent values. We will assume that each bidder has only an estimate of the value, and we will call this estimate a signal. For example, bidders who compete for the right to exploit oil from some area might have different estimates of the amount of oil in the soil and the value of the right depends on the signals of all bidders. It is natural to assume that these signals are postitively correlated. In fact we will assume that signals are affiliated which means that a high value of one bidder's signal makes it more likely that other bidders have high signals as well. We will assume that bidder *i*'s signal S_i is distributed on some interval $[0,\omega]$. The value of the object for bidder *i* V_i is a function of signals of all *N* bidders $V_i = v_i(S_1, S_2, ..., S_N)$, where v_i is bidder *i*'s valuation function which is increasing in all *N* signals. When valuation functions for bidders are identical we have the case of *common values* $V = v(S_1, S_2, ..., S_N)$. For example, if bidders compete for an object which they plan to resell in the future, the resale price represents the common value. Even though this value is the same for all bidders they only have some estimate of that value. Each bidder bids according to a bidding function $b_i = b(S_i)$ which maps his signal into a bid.

In the majority of models in this paper we will assume that bidders are symmetric. In the case of interdependent values there are *two aspects of symmetry*. The first aspect requires that bidders' signals are distributed according to the same distribution function on the same support. The second aspect concerns the symmetry of the valuation functions. For example, if there are three bidders, and value functions are symmetric, they might have the following form:

$$V_1 = S_1 + \frac{1}{2}S_2 + \frac{1}{2}S_3\,,\; V_2 = S_2 + \frac{1}{2}S_1 + \frac{1}{2}S_3\,,\; V_3 = S_3 + \frac{1}{2}S_1 + \frac{1}{2}S_2 \tag{1}$$

The symmetry of the value functions means that the signals of bidders 2 and 3 can be interchanged without affecting the value of bidder 1, and the same holds for other bidders. In other words, if there are N bidders the value function for bidder i is symmetric if it is symmetric in all N-1 signals of his competitors. In the common value model the value function is the same for all bidders. For example, when there are three bidders the value function might have the following form, which is the same for all bidders $V = S_1 + S_2 + S_3$.

When one of the two above assumptions is not met we have the asymmetric case. For example, if valuations are such that:

$$V_1 = S_1 + \frac{1}{3}S_2 + \frac{1}{2}S_3\,,\; V_2 = S_2 + \frac{1}{3}S_1 + \frac{1}{2}S_3\,,\; V_3 = S_3 + \frac{1}{3}S_1 + \frac{1}{2}S_2\,, \tag{2}$$

we are dealing with the asymmetric case. In this case if for one bidder the signals of his rivals are interchanged, his value will change.

One more problem arises in the interdependent value case that was not present in the private value case. This is the phenomenon of the *winner's curse*. We will show below that a bidding strategy of each bidder is increasing in his signal and the bidder with the highest signal wins the auction. But after that bidder wins he might discover that the value of the object is less than his bid and this phenomenon represents the winner's curse. The bidder who wins has the most optimistic estimate of the value and winning the auction brings bad news that other bidders have lower signals. In other words, the winning bidder's conditional estimate of the value upon winning is lower than his conditional estimate before the auction ends[2]. In order to avoid the winner's curse each bidder has to shade his bid. The winner's curse is more severe in common value auctions than in auctions with interdependent values.

We have mentioned the natural assumption that in the interdependent value case bidders' signals are affiliated. This term was coined by Milgrom and Weber (1982) in their seminal paper. Affiliation means that if one bidder has a high signal, then it is more likely that other bidders have high signals as well. We will define this term more formally. Suppose that signals S_1 and S_2 are affiliated and take two realizations of these signals such that $s_1' > s_1$ and $s_2' > s_2$ and denote by $f(\cdot)$ the joint density of the two signals. Affiliation implies that it is more likely that both signals have high or low value than that one signal has a high and other a low value. Formally, affiliation implies that $f(s_1', s_2) \cdot f(s_1, s_2') \le f(s_1, s_2) \cdot f(s_1', s_2')$.

As we will see soon, bidding strategies in this environment are more complicated than in the private value case, because each bidder has to condition his bid on his signal and on the signals of other bidders. In sealed-bid auctions bidders cannot obtain the information about signals of other bidders, so each bidder's best estimate is that other bidders have received the same signal as he obtained. The same holds for a Dutch auction which ends when one bidder accepts the price. In the private value case, the English and a second-price auction are strategically equivalent. However in English auction with interdependent values bidders drop one by one from the auction and the remaining bidders obtain valuable information about the signals of non-active bidders by observing the prices at which they drop out. Because of this effect English auction is no longer strategically equivalent to a second-price auction. On the other hand a Dutch and a first-price auction continue to be strategically equivalent because in both the signal inference is impossible.

2 Milgrom (1981*b*) discusses in more detail the impact of bad news on the conditional expectation of the value.

The rest of the paper is organized as follows. In the second part we derive equilibrium bidding strategies in second-price, first-price, and English auctions. In the third part we will see how some auction forms can be ranked in terms of expected revenue accruing to the seller and how the seller can affect his expected revenue by revealing his private information. The fourth part deals with reserve prices and entry fees. In the fifth part we analyse some conditions that are needed for an English auction to allocate efficiently. In the sixth part we analyse common value auctions. The last section concludes the discussion.

2. EQUILIBRIUM BIDDING STRATEGIES

In this section we will derive bidding strategies in second-price, first-price, and English auctions, and we will see that the strategies in a second-price and an English auction are quite different. We will show that these strategies take into account the effect of the winner's curse.

Second-price auction

In a second-price auction bidders submit bids in sealed envelopes. The bidder who has submitted the highest bid wins and he pays the price that is equal to the second highest bid. Equilibrium bidding strategies in a second-price auction are derived in Milgrom (1981*a*) and in Milgrom and Weber (1982). Consider bidder 1 who has a signal $S_1 = s$ and suppose that the highest signal of other bidders is $Y_1 \equiv \max_{i \neq 1} S_i$. Define the following function $v(s, y) = E[V_1 | S_1 = s, Y_1 = y]$. This function represents the conditional expected value of bidder 1 when he has a signal $S_1 = s$ and the highest signal of other bidders is y. This conditional expectation is strictly increasing in the arguments[3] s, y. We argue that bidder 1 will bid according to the symmetric equilibrium strategy $b(s) = v(s, s)$, which means that he will assume that the the highest signal of his rivals is equal to his signal. In order to prove this result, suppose that other bidders follow conjectured bidding strategies and we will prove that bidder 1 does not have an incentive to deviate from this strategy. Bidder 1 will win the auction if $b(s) > b(y_1)$. Denote the optimal bid of bidder 1

[3] A formal proof of this statement can be found in Menezes and Monteiro (2004) p. 62-63 and in Milgrom (1981*a*).

as $b = b(s)$. If he wins the auction, he pays the second highest bid. The bidding function $b(s) = v(s,s)$ is strictly increasing, because the conditional expectation of the value is increasing in s, and we can find the inverse of the bidding function $s = b^{-1}(b)$. The expected profit of bidder 1, who has a signal s, bids b, and pays $b(y)$, whereas his competitors follow the proposed strategy, is:

$$\Pi(b,s) = \int_0^{b^{-1}(b)} (v(s,y) - b(y))g(y\,|\,s)dy = \int_0^{b^{-1}(b)} (v(s,y) - v(y,y))g(y\,|\,s)dy\,, \tag{3}$$

where $g(y\,|\,s)$ is the conditional density of Y_1 given $S_1 = s$ and the second equality follows from the fact that the bidder with the second highest bid follows the conjectured equilibrium strategy $b(y) = v(y,y)$. Since the value function is increasing in s, for $s > y$, $v(s,y) - v(y,y) > 0$ and for $s < y$, $v(s,y) - v(y,y) < 0$ and the integral is maximized for $b^{-1}(b) = s$. Thus, if other bidders follow the proposed strategy, bidder 1 cannot do better than to follow the strategy $b(s) = v(s,s)$, and since the same argument holds for each bidder these strategies constitute a Nash equilibrium.

The intuition behind the equilibrium strategy is as follows. Bidder 1 who has a signal s bids an amount $b(s) = v(s,s)$ that would make him indifferent to winning or losing if the bidder with the second highest bid has the same signal $Y_1 = s$. In that case the winner would be determined by flipping a coin and bidder 1 would infer that the other bidder has a signal $Y_1 = s$ and the value of the object for him after learning the rival's signal would be $v(s,s) = E[V_1\,|\,S_1 = s, Y_1 = s]$. Since his rival followed the strategy $b(s) = v(s,s)$, bidder 1 would pay a price $v(s,s)$ upon winning and his *ex-post* profit would be equal to zero. Thus by following the equilibrium strategy bidder 1 will just break even at worst, and he can obtain a positive profit if the second highest signal is lower than his signal. In the latter case $s > y$ and bidder 1's *ex post* value would be $v(s,y)$ and he would obtain a positive profit $v(s,y) - v(y,y) > 0$, because his competitor bids $b(y) = v(y,y)$ which represents the price that bidder 1 pays.

Milgrom (1981*a*) noted that the above equilibrium is an ex post equilibrium when there are two bidders, but the argument is more extensively explained in

Milgrom (2004). In an *ex post* equilibrium a bidder does not regret using his strategy $v(s,s)$ even after learning the signals of other bidders. Suppose that bidder 1 has a signal s and that he learns that the second highest signal is y and that the second highest bid is $b(y)=v(y,y)$. After learning that signal, bidder 1 updates the value for the object at $v(s,y)$ and he maximizes the following profit function:

$$\Pi = \max_b [v(s,y) - v(y,y)] 1_{b>b(y)}, \tag{4}$$

where $1_{b>b(y)}$ is an indicator function that has a value 1 if $b>b(y)$ and 0 otherwise. If $s>y$, then any bid $b>b(y)$ maximizes bidder 1's profit, including the bid $b(s)>b(y)$, because *b* only affects the indicator function and $b(\cdot)$ is an increasing function. Similarly, if $s<y$, then any bid $b<b(y)$ maximizes the profit function, including the bid $b(s)<b(y)$. Thus, even after learning his rival's type, bidder 1 does not regret using his previous strategy, i.e. he does not want to change his strategy. The same argument applies to bidder 2, and the equilibrium of the second-price auction is an *ex post* equilibrium because no bidder will change his strategy even after learning the other bidder's signal. The equilibrium has the property that the equilibrium bidding strategy depends only on the bidder's signal and not on the signals of his competitors. Therefore bidders do not have an incentive to expend resources to determine the other bidder's signal and a second-price auction with two bidders is strategically simpler than a first-price auction, where each bidder could benefit by knowing his competitors' signals. Unfortunately, once there are more than two bidders, they might regret using the previously described strategies, which means that the equilibrium of the second-price auction might not be an ex post equilibrium.

Harstad and Levin (1985) study a special class of auction in which the highest signal represents a sufficient statistic for all other signals. In other words the highest signal gives the same information about the value of the object as the entire vector of signals. Denote by s the highest signal and by y the second highest signal. In this case for any y and y' such that $s>y$ and $s>y'$ the following equality holds $v(s,y)=v(s,y')$. This implies that only the highest signal is important for the valuation of the object. Harstad and Levin (1985) show that for this class of auction the Nash equilibrium in bidding strategies discussed above represents an equilibrium in dominant strategies. In other words, a bidder with signal s cannot benefit from bidding lower than $v(s,s)$ nor by bidding higher than $v(s,s)$ and bidding $v(s,s)$ represents the dominant strategy. It is important to note that this is a very restrictive class of auction.

First-price auction

Wilson (1969, 1977) was the first to study equilibrium bidding strategies in first-price auctions. However we will follow the approach of Milgrom and Weber (1982), which is more intuitive. Denote by $G(y \mid s)$ the conditional distribution of $Y_1 \equiv \max_{i \neq 1} S_i$ given $S_1 = s$ and by $g(y \mid s)$ the corresponding density. Suppose that bidder 1 has a signal s and suppose that other bidders follow the increasing strategy $b(\cdot)$. The expected profit of bidder 1 with a signal $S_1 = s$ who bids as if his signal is $S_1 = z$ is:

$$\Pi(z,s) = \int_0^z (v(s,y) - b(z))g(y \mid s)dy = \int_0^z v(s,y)g(y \mid s)dy - b(z)G(z \mid s) . \tag{5}$$

By using Leibnitz's rule for differentiating integrals, we obtain the following first order condition:

$$(v(s,z) - b(z))g(z \mid s) - b'(z)G(z \mid s) = 0 . \tag{6}$$

It is not difficult to prove that in equilibrium bidder 1 will find it optimal to set $z = s$, but we omit the proof. By setting $z = s$, the first order condition becomes:

$$b'(s) = (v(s,s) - b(s))\frac{g(s \mid s)}{G(s \mid s)} . \tag{7}$$

By solving this differential equation, we obtain the equilibrium bidding strategy. Note that it is necesarry that $v(s,s) - b(s) \geq 0$, since otherwise the expected profit would be negative and the bidder would be better off by bidding 0. The solution to this differential equation by using the method of the integrating factor is given in appendix A, and the equilibrium bidding strategy can be written as:

$$b^*(s) = \int_0^s v(y, y) dL(y \mid s) , \tag{8}$$

where

$$L(y \mid s) = \exp\left(-\int_y^s \frac{g(t \mid t)}{G(t \mid t)} dt \right) . \tag{9}$$

Integrating (8) by parts, we obtain an alternative form of the bidding strategy:

$$b^*(s) = v(s, s) - \int_0^s L(y \mid s) dv(y, y) . \tag{10}$$

The last expression is more intuitive, because it tells us that a bidder in a first-price auction bids lower than in a second-price auction, where his bid is $b^*(s) = v(s, s)$. This result is intuitive, since in a first-price auction a bidder pays his bid and in a second-price auction he pays the second highest bid, and he places a lower bid in a first-price auction because his bid determines his payment. Thus the integral in (10) could be considered as bid-shading, i.e. the amount by which a bidder lowers his bid relative to his estimate of the value.

One important property of the bidding strategy is worth noting. It can be shown that the equilibrium bid is less than the expected value conditional on winning, because a bidder wants to avoid the *winner's curse*. The winner's curse is not present in private value auctions, but is very important in auctions with interdependent values. The bidder who wins the auction has the highest signal, and after winning the auction he learns that other bidders have lower signals. Since values are affiliated, his *ex post* value conditional on winning is lower than his conditional estimate of the value before the auction ends. In other words, the bidder who wins the auction was over optimistic and it might happen that he payed more than the *ex post* value of the object, and if this is the case he suffers the winner's curse. The severity of the winner's curse increases with the number of bidders, because if a bidder wins the auction with several competitors he can conclude that he was too optimistic, but if he wins the auction with 100 bidders he can conclude that he was extremly optimistic. In order to avoid the winner's curse a bidder places a bid that is lower than the expected value conditional on winning both in first-price and second-price auctions. This result is formally proved in appendix B.

English auction

In deriving equilibrium bidding strategies in an English auction we will in fact use a so-called Japanese version of the English auction. In this auction the price is continuously raised on an electronic display and bidders show their willingness to buy by pressing a button. When a bidder releases the button he drops out from the auction and he cannot re-enter the auction later on. The current price, the number of active bidders, and the prices at which non-active bidders dropped out are commonly known. The last bidder who stays in is the winner and he pays the price at which the next-to-last bidder dropped out.

Since signals are affiliated the additional information obtained during the auction influences the active bidders' values. In other words, when one bidder drops out the other bidders can infer his signal and update their estimates of the value. This implies that English and second-price auctions are no longer strategically equivalent, because signal inference is impossible in a sealed-bid second-price auction. These two auctions are equivalent when there are two bidders because when one bidder drops out the auction is over, or when the number of active bidders and prices at which non-active bidders have dropped out from English auction are not revealed.

The strategy of a bidder in an English auction is to determine the price at which he will drop out. His strategy is a function of his signal, the number of active bidders, and the prices at which non-active bidders have quit. Suppose that there are N bidders and denote by k the number of active bidders, by $p_{k+1} \geq p_{k+2} \geq ... \geq p_N$ the prices at which the N-K bidders have dropped out, and by s the signal of an active bidder. Therefore the strategy of an active bidder is to determine the price at which he will drop out $b^k(s, p_{k+1},... p_N)$.

When all N bidders are active the strategy for each bidder is to drop out when the price reaches his conditional expectation of the value. At this stage the signals of other bidders are not known to a particular bidder and he assumes that his competitors have the same signal. Consider bidder 1 who has the highest signal $S_1 = s$ and denote by $Y_1, Y_2, ..., Y_{N-1}$ the highest, second highest, and so on, signal of his competitors. Therefore, when all N bidders are active, the bidding strategy of bidder 1 is:

$$b^N(s) = E[V_1 | S_1 = s, Y_1 = s, ..., Y_{N-1} = s]. \quad (11)$$

Suppose that bidder N with a signal s_N drops out when the price on the electronic display reaches p_N. The conditional expectation is strictly increasing in its first argument, and the bidding strategy is increasing in the bidder's signal as well. Since $b^N(s_N)$ is strictly increasing in s_N, there exists a unique s_N, such that $p_N = b^N(s_N)$, which means that the remaining bidders can infer the signal of the N-th bidder who has dropped out by inverting the bidding function. Now the remaining N-1 bidders update their estimates of the value and bidder 1 follows the strategy:

$$b^{N-1}(s) = E[V_1 | S_1 = s, Y_1 = s, ..., Y_{N-2} = s, Y_{N-1} = s_N]. \quad (12)$$

The price is then raised on the electronic display, and suppose that bidder N-1 with a signal s_{N-1} drops out when the price reaches $p_{N-1} \geq p_N$. The remaining N-2 bidders can infer his signal and update their estimates of the value. The equilibrium bidding strategy of bidder 1 now becomes:

$$b^{N-2}(s) = E[V_1 | S_1 = s, Y_1 = s, ..., Y_{N-3} = s, Y_{N-2} = s_{N-1}, Y_{N-1} = s_N]. \quad (13)$$

Proceeding in this way we can conclude that after N, N-1,...,k+1 bidders have dropped out at prices $p_{k+1} \geq p_{k+2} \geq ... \geq p_N$, the bidding strategy of bidder 1 when there are k active bidders becomes:

$$b^k(s) = E[V_1 | S_1 = s, Y_1 = s, ..., Y_k = s, Y_{k+1} = s_{k+2}, ..., Y_{N-1} = s_N]. \quad (14)$$

Finally, when only two bidders remain they in fact engage in a second-price auction and bidder 1's strategy is:

$$b^2(s) = E[V_1 \mid S_1 = s, Y_1 = s, Y_2 = s_3, ..., Y_{N-1} = s_N]. \tag{15}$$

Since bidder 1 has the highest signal, bidder 2 drops out and bidder 1 wins. Bidder 1 can infer that bidder 2's signal is $Y_1 = s_2$, and his *ex post* expected value becomes $E[V_1 \mid S_1 = s, Y_1 = s_2, Y_2 = s_3, ..., Y_{N-1} = s_N]$. Bidder 2 dropped out at the price $b^2(s_2) = E[V_2 \mid S_1 = s_2, Y_1 = s_2, Y_2 = s_3, ..., Y_{N-1} = s_N]$ and bidder 1 pays exactly that price, so the expected profit of the winning bidder is:

$$\Pi(s) = E[V_1 \mid S_1 = s, Y_1 = s_2, Y_2 = s_3, ..., Y_{N-1} = s_N] - E[V_2 \mid S_1 = s_2, Y_1 = s_2, Y_2 = s_3, ..., Y_{N-1} = s_N]. \tag{16}$$

To verify that the strategies described above represent equilibrium strategies, note that the expected profit is positive if $s > s_2$ and since bidder 1 cannot affect the price he pays, he cannot do better than to follow the proposed strategies. On the other hand if $s < s_2$, but bidder 1 decides to stay in after the price has reached his estimate of the value, his profit would be negative. Thus he cannot do better than to follow the proposed strategies.

We have seen that the bidding strategy is increasing in the bidder's signal and when bidders are symmetric they drop out in order of their signals, which means that the bidder with the lowest signal drops out first, followed by the bidder with the second lowest signal, and so on. The bidder with the highest signal is the winner. However when bidders are asymmetric they might not need to drop out in the increasing order of their signals. Furthermore the bidder with the highest signal need not be the bidder with the highest value, and the allocation might be inefficient even when bidders are symmetric. We will examine these problems in more detail below.

The intuition behind the bidding strategies can be understood in the following way. Suppose that all bidders are active and that bidder 1 follows the strategy $b^N(s) = E[V_1 \mid S_1 = s, Y_1 = s, ..., Y_{N-1} = s]$. In determining this strategy, bidder 1 asks himself will he be happy if all other bidders drop out instantly. If this happens, he infers that all other bidders have the same signal

$Y_1 = Y_2 = ... = Y_{N-1} = y$. The *ex post* value for bidder 1 would be $E[V_1 | S_1 = s, Y_1 = y, ..., Y_{N-1} = y]$ and he would pay the price equal to $E[V_i | S_1 = y, Y_1 = y, ..., Y_{N-1} = y]$. His expected profit is positive if $s > y$ and negative if $s < y$ and equal to zero when all other bidders have the same signal as bidder 1, i.e. when $s = y$. Thus, the bidding strategy demands that the bidder stay active until the price reaches the point when he is merely indifferent to winning or losing at that price. By following this strategy he can obtain a profit of zero at worst.

Since all signals are revealed in the course of the auction the winning bidder does not regret winning. On the other hand a losing bidder does not regret losing, because if he had won the auction it would have been at a price that was higher than his *ex post* expected value of the object. Thus bidders would not change their behaviour even if all signals were common knowledge, and the equilibrium of the English auction is an ex post equilibrium.

3. THE LINKAGE PRINCIPLE

The linkage principle refers to the fact that the seller's expected revenue rises when the price paid by the winning bidder is more closely linked to his signal. This effect stems from the fact that signals are affiliated. Therefore auctions which lead to revelation of signals during the course of the auction, or have the property that the price paid by the winning bidder is more closely related to his signal, lead to a higher expected revenue for the seller, because he can extract more surplus from the winning bidder. The same holds for the seller's information. Disclosing that information could be to the seller's benefit. Milgrom (1989) synthesizes the linkage principle in the following way. If the price paid by the winning bidder could be linked to the signals of other bidders or the seller, which are affiliated with the winning bidder's signal, then the winning bidder in such an auction is made worse off and the seller better off.

Expected revenue ranking

As we have explained, due to the linkage principle auctions that reveal other bidders' signals during the course of the auction lead to a higher expected revenue for the seller. We have seen that in English auction a bidder specifies the price at which he will drop out. When one bidder drops out the remaining bidders infer his signal and update their values. Therefore the value that the winning bidder attaches to the object depends on all signals. The intuition underlying this result is that additional information about competitors' signals obtained during the auction weakens the winner's curse and a bidder bids more aggressively. On the other hand the signal inference is impossible in a second-price auction and a bidder will bid more cautiously than in an English auction in order to avoid the winner's curse. This result shows that the expected revenue of the seller is higher in an English than in a second-price auction, and we have shown above that the equilibrium strategies in these two auctions are different. Thus an English and a second-price auction are equivalent in the weak sense, i.e. they are equivalent only with private values, and in the case of interdependent values they are equivalent only when there are two bidders, because when one bidder drops out the auction is over and the winning bidder cannot benefit from inferring his rival's signal. When there are more than two bidders the equivalence fails and the English auction gives higher expected revenue to the seller than a second-price auction.

Note that the Dutch and the first-price auction are equivalent even with interdependent values, because the Dutch auction is over when one bidder accepts the price and other bidders cannot benefit from inferring his signal. The same holds for a first-price auction, which by its very nature makes the signal inference impossible. Since in both auctions the winning bidder pays his bid he faces the same problem, and his strategy in these two auctions is the same. This result implies that the Dutch and the first-price auction are equivalent in the strong sense, i.e. they are equivalent with both private and interdependent values. Thus the expected revenue of the seller is the same in a Dutch and a first-price auction.

Finally we have to compare the expected revenue in a first-price and a second-price auction. In a first-price auction the winning bidder pays his bid which depends only on his signal. On the other hand the expected price paid by the winning bidder in a second-price auction depends on his signal and the second highest signal. Klemperer (2004) gives the intuitive explanation of why a second-price auction yields higher expected revenue than a first-price auction. The winning bidder's profit stems from his private information (his signal). When the price paid depends on the other bidders' information it is more closely related to the winning bidder's information because signals are affiliated, which reduces his expected profit. Since the seller's expected revenue comes at the expense of the expected profit of the winning bidder, this reduction of the winning bidder's expected profit leads to a higher expected revenue for the seller. An alternative explanation is that the winner's curse is reduced for the winning bidder since he pays the second highest bid, which makes him bid less cautiously. We have provided the formal proof of the expected revenue ranking between a second-price and a first-price auction in appendix B.

Now we can summarize our findings. We have shown that with interdependent values the English auction leads to a higher expected revenue accruing to the seller than in a second-price auction, which in turn leads to a higher expected revenue than in a first-price or Dutch auction. Milgrom (1989) argues that the expected revenue ranking might explain the dominant use of the English auction in practice.

War of attrition and all-pay auctions

Krishna and Morgan (1997) derive equilibrium bidding strategies in the two special forms of sealed-bid auctions: war of attrition and all-pay auctions. In an all-pay auction bidders submit bids in sealed envelopes and the bidder with the highest bid obtains the item, but all bidders pay their bid. In the war of attrition all bidders pay their bid except the winning bidder, who pays the second-highest bid. Thus the all-pay auction could be referred to as a first-price all-pay auction because of its analogy with a first-price auction, whereas the war of attrition could be referred to as a second-price all-pay auction, because it has some resemblance to the second-price auction. For example, lobbying activities could be considered as all-pay auctions, because all agents pay something and only the agent who has provided the highest amount can achieve his objectives in lobbying activities. The other example is a patent race, when all companies have expenditures in research and developement and only the most successful company will realise a profit from its research. Krishna and Morgan (1997) show that with affiliated signals the war of attrition yields higher expected revenue to the seller than a second-price auction (see appendix C). The same holds for an all-pay auction which yields higher expected revenue to the seller than a first-price auction (see appendix D).

The intuition for this result is as follows. Recall that a second-price auction yields higher expected revenue than a first-price auction because the price paid by the winning bidder is more closely linked to his signal, which reduces his expected profit and increases the seller's expected revenue. This effect could be called the *second-price effect*. On the other hand in a war of attrition and an all-pay auction all bidders pay a positive amount. When signals are affiliated each bidder concludes that if he has a high signal it is more probable that other bidders have high signals as well, and this raises the probability that he will lose the auction and that he will have to pay his bid without receiving the item. In order to avoid this outcome each bidder bids higher in an all-pay auction and a war of attrition than he bids in an auction in which only the winner pays. This bid increase could be considered as the *losing bid effect*. Therefore an all-pay auction out-performs a first-price auction because of the losing bid effect. On the other hand the war of attrition out-performs a second-price auction because it includes both effects, while a second-price auction includes only the second-price effect. Furthermore the war of attrition out-performs first-price, second-price, and all-pay auctions because it includes both effects, while second-price and all-pay auctions include only one effect and first-price auctions do not include either effect. Finally, an all-pay auction cannot be ranked with a second-price auction, because each comprises one effect and it is not clear in advance which effect will dominate.

We can conclude that the war of attrition yields the highest expected revenue of all four sealed-bid auctions, an all-pay auction yields higher expected revenue than a first-price auction, and the comparison between all-pay and second-price auctions is ambiguous.

One might wonder why all-pay auctions and the war of attrition are rarely seen in practice, despite the fact that these two forms turn out to be revenue superior than standard sealed-bid auctions. One possible explanation is that the competition from other sellers might preclude the use of these two auction forms.

Informed seller

Now assume that the seller possesses information that is valuable to bidders. It could be for example an expert's estimate of the value of the object. Denote the seller's signal by $\hat{S}$ and suppose that $\hat{S}$ is affiliated with other bidders' signals. Each bidder now takes this signal into consideration, and his valuation function becomes $V_i = v_i(\hat{S}, S_1, S_2, ..., S_N)$, where the value function is symmetric in the last N-1 arguments.

The same principle described in the expected revenue ranking applies here. The winning bidder's profit decreases when the price is more highly linked to his information. Since the seller's signal is affiliated with the winning bidder's signal, revealing that information links more strongly the price paid with the wining bidder's information, and reduces his expected profit. Thus the optimal policy of the seller who possesses information about the object is to reveal that information, since this strategy increases his expected revenue. Milgrom and Webber (1982) formally prove that in English, first-price, and second-price auctions the seller's expected revenue increases when he reveals his private information. The result relies on the law of iterated expectations and is not difficult to prove. However we omit the proof in this paper. In addition Milgrom and Webber (1982) prove that the seller should always reveal his information and that censoring information is useless. For example, if the seller reveals his signal $\hat{S}$ when it is higher than some threshold value, the absence of signal revelation will be interpreted by bidders as a bad signal.

This result about the impact of public information on the seller's expected revenue could be applied to the expected revenue ranking of an English and a second-price auction. Recall that in a symmetric English auction bidders drop one by one in the increasing order of their signals. The last two remaining bidders actually engage in a second-price auction, and the preceding part of the English auction could be considered as a stage when public information is released. Since we know that the release of public information leads to a higher expected revenue for the seller, the English auction, which incorporates a stage of information revelation before the last two bidders compete as in a second-price auction, will yield higher expected revenue for the seller than an ordinary second-price auction.

Perry and Reny (1999) provide an example in which the seller's expected revenue in a second-price auction with two asymmetric bidders decreases when he releases his information. The intuition behind their result is as follows. In a second-price auction the losing bids are based on underestimates of the other bidders' signals. On the other hand the winning bid is based on an overestimate of the other bidders' signals. When the seller reveals his information, on average the losing bidders become more confident and they increase their bids. On the other hand the winning bidder becomes more cautious, and on average decreases his bid. In the case of two bidders, if the previously winning bid falls below the previously losing bid, the revenue of the seller decreases when he reveals his information in a second-price auction.

Failures of the linkage principle

It is important to note that the expected revenue ranking relies on the assumption that bidders are symmetric. At the begining of our paper we explained that in auctions with interdependent values there are two aspects of symmetry. First, bidders' signals have to be distributed according to the same density function at the same interval. Second, a bidder's value function has to be symmetric in the signals of other bidders, meaning that if the signals of other bidders are interchanged the value remains the same. When one of the aspects of symmetry is relaxed the expected revenue ranking result might not hold. For example, it might happen that the expected revenue in a second-price auction is higher than in an English auction. Hence, similarly to the case of private values where the symmetry was crucial for the revenue equivalence, in the case of the interdependent values the symmetry is crucial for the expected revenue ranking.

Perry and Reny (1999) established that the predominant use of the English auction is based on its superiority in terms of the seller's expected revenue when the values are interdependent. On the other hand in our previous paper we have shown that second-price and English auctions are more vulnerable to collusive behaviour than first-price auctions. Since objects with interdependent values are more frequently auctioned than objects with private values the negative effect of collusive behaviour seems to be outweighed by the linkage principle. Perry and Reny (1999) prove by means of a counter example why the linkage principle does not extend from single object auctions to multiple object auctions. Their argument is beyond the scope of this paper, since it deals with multiple object auctions. However this result shows us that we should be cautious when extrapolating conclusions from single object to multiple object auctions.

4. RESERVE PRICES AND ENTRY FEES

The reserve price is a price below which the seller is not willing to sell the object. We have shown in the private value case that the reserve price increases the seller's expected revenue. In a second-price auction the gain stems from the event that the reserve price is higher than the second-highest value. On the other hand if the reserve price is higher than the highest value the object remains unsold and the seller suffers a loss. For a small reserve price the expected gain outweighs the expected loss and the seller's expected revenue increases when he posts a reserve price. However when signals are affiliated the highest and second-highest values are closer to each other than with private values and the event that the reserve price is higher than the second highest value and lower than the highest value is less likely to occur. Hence, while the expected loss remains the same as in the private value case, the expected gain is lower. When the affiliation is too strong the expected gain can outweigh the expected loss and the reserve price reduces the seller's expected revenue.

We have seen that with private values the optimal reserve price is independent of the number of bidders and the seller posts a reserve price that is higher than his value for the object. Levin and Smith (1996) show that with affiliated signals, as the number of bidders increases the optimal reserve price converges to the seller's value for the object. The intuition for this result is that with affiliated signals, as the number of bidders increases the difference between the highest and second-highest value becomes infinitesimal, and the expected gain from posting a reserve price that is higher than the seller's value tends to zero. Thus while the expected loss is positive the expected gain tends to zero, and the seller lowers the reserve price towards his value. If the seller's value is zero, then it is clear that it is not optimal to set the reserve price. We have seen in the private value case that when the seller sets a higher reserve price than his value, it is possible that the highest value is lower than the reserve price while it is higher than the seller's value, which means that the seller retains the item even though he does not have the highest value. Clearly this allocation is inefficient. However when values are interdependent, the possibility that the outcome of the auction with the reserve price is inefficient decreases as the number of bidders increases, and at the limit when the number of bidders tends to infinity the probability of inefficient allocation tends to zero, since the reserve price tends to the seller's value. This suggests that with interdependent values competition enchances efficiency as in ordinary markets.

An entry fee represents an amount that a bidder has to pay to participate in the auction. Milgrom and Weber (1982) prove that when the seller uses both reserve prices and entry fees, the strategy of setting high entry fees and low reserve prices leads to higher expected revenue accruing to the seller. More precisely, Milgrom and Weber (1982) define the *screening level* as the lowest possible signal, s^*, that would make a bidder participate in the auction. If a bidder's signal exceeds this threshhold level s^* a bidder will participate in the auction, otherwise he will not participate. They show that the two combinations of different reserve prices and entry fees that lead to the same screening level yield different expected revenues to the seller. In particular the auction with high entry fees and low reserve prices outperforms the auction with low entry fees and high reserve prices. This result is formally proved in appendix E.

One particular problem that might arise in the presence of entry fees is that the auction is not regular, meaning that some bidders with a signal higher than s^* will find it optimal not to participate. This problem complicates the analysis significantly.

5. THE EFFICIENCY OF ENGLISH AUCTIONS WITH ASYMMETRIC BIDDERS

An allocation is efficient if the bidder with the highest value obtains the object. Formally, an allocation is efficient if for all signals $(s_1,..., s_N)$, bidder *i* gets the object when the following inequality holds $v_i(s_1,..., s_N) > v_j(s_1,..., s_N)$, for all *j*. In the previous discussion we have shown that the bidder with the highest signal obtains the object in the English auction. In this part of the paper we study whether English auction results in an efficient allocation. The general answer is no, even when bidders are symmetric. In other words the bidder with the highest signal need not be the bidder with the highest value. For example, if there are two symmetric bidders with signals s_1 and s_2 and value functions:

$$v_1 = \frac{1}{4}s_1 + \frac{3}{4}s_2 \text{ and } v_2 = \frac{3}{4}s_1 + \frac{1}{4}s_2, \tag{17}$$

and if $s_1 > s_2$, then $v_2 > v_1$. Thus the bidder with the highest signal has a lower value and the allocation is clearly inefficient. What went wrong in this example? The problem is that each bidder's signal has a higher influence on the value of the other bidder than on bidder's own value. When each bidder's signal has a higher impact on his value than on the other bidder's value, the allocation with symmetric bidders is efficient. We will refer to this condition as the *single crossing condition*.

When bidders are asymmetric the problem with efficiency can be even worse. We will illustrate this problem with the following two examples concerning first-price and second-price auctions, taken from Dasgupta and Maskin (2000).

Suppose that there are two bidders in a first-price auction and that the first bidder has a signal s_1 distributed on $[0,1]$ and the second bidder has a signal s_2 distributed on $[0,10]$. In this auction it must be the case that $b_1(1) = b_2(10)$. To see why, suppose that $b_1(1) < b_2(10)$, then bidder 2 with value 10 can reduce his bid towards $b_1(1)$ and still win with probability 1. But, when $b_1(1) = b_2(10)$, bidder 2 with a value slightly less than 10, will bid less then $b_1(1)$ and will lose the auction, even though he has a higher value.

Consider the following example that deals with a second-price auction. Suppose that there are three bidders with value functions:

$$v_1 = s_1 + \frac{1}{2}s_2 + \frac{1}{4}s_3\,,\; v_2 = s_2 + \frac{1}{4}s_1 + \frac{1}{2}s_3 \text{ and } v_3 = s_3\,. \tag{18}$$

Suppose that $s_1 = s_2 = 1$. If for some small ε, s_3 is equal to $s_3 = 1 + \varepsilon$, then $v_1 > v_2 > v_3$. But if $s_3 = 1 - \varepsilon$, then $v_2 > v_1 > v_3$. Hence the efficient allocation between bidders 1 and 2 depends on the signal of bidder 3. But, as we have seen from the previous discussion, a bidder submits a bid in a second-price auction that depends only on his signal and in this example the efficient allocation between bidders 1 and 2 depends on bidder 3's signal, which is clearly inefficient.

Dasgupta and Maskin (2000) prove that when the bidders' value functions satisfy the single crossing condition, the allocation is efficient. For all *i* and *j*, the single crossing condition can be defined in the following way:

$$\frac{\partial v_i}{\partial s_i} > \frac{\partial v_j}{\partial s_i}\,, \tag{19}$$

when $v_i = v_j = \max_k v_k$. The single crossing condition implies that when the values of the bidders *i* and *j* are equal and maximal, the signal of bidder *i* has a higher impact on his value than on the value of bidder *j*.

The single crossing condition is not sufficient to guarantee efficiency when there are three or more bidders. Krishna (2003) shows that in this case the *average crossing condition* is necessary to achieve efficiency. This condition requires that the impact of the signal of each bidder on the average value is higher than the impact of that signal on some other's value. If there are *N* bidders, the average value is defined as:

$$\bar{v} = \frac{1}{N}\sum_i v_i\,. \tag{20}$$

Denote by $\overline{v}_i \equiv \dfrac{\partial \overline{v}}{\partial s_i}$ the derivative of the average value with respect to s_i and by $v_{ji} \equiv \dfrac{\partial v_j}{\partial s_i}$ the derivative of bidder *j*'s value with respect to s_i. For any vector of signals $\mathbf{s}$, such that values of all *N* bidders are equal and maximal, the average crossing condition requires that the following inequality is satisfied:

$$\overline{v}_i > v_{ji}\,. \tag{21}$$

The intuition behind the average crossing condition is that the impact of bidder *i*'s signal on his value is higher than the impact on the average value, since the impact of *i*'s signal on values of other bidders must be below average.

We can define the single and average crossing condition by using the so-called influence matrix, in which *i,j* entry represents the influence of *j*'s signal on *i*'s value. For example, when there are three bidders the influence matrix has the following form:

$$\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}, \tag{22}$$

where v_{11} represents the impact of bidder 1's signal on his value, v_{12} the impact of bidder 2's signal on bidder 1's value, and so on.

The single crossing condition requires that any element on the main diagonal is higher than any off-diagonal element. The average crossing condition requires that any element on the main diagonal is larger than the average of the elements in that column. For example, for bidder 1 this condition is satisfied if $v_{11} > \overline{v}_1 = (v_{11} + v_{21} + v_{31})/3$. Since the impact of some bidder's signal on his value is larger than the impact of his signal on some other bidder's value, the average crossing condition implies that $v_{11} > \overline{v}_1 > v_{21}$ and $v_{11} > \overline{v}_1 > v_{31}$.

Finally Krishna (2003) shows that if values satisfy the *cyclical crossing condition*, the English auction is efficient. This condition requires that for any two bidders i and $i+1$, every signal s_j, with the exception of s_{i+1}, has a greater influence on v_i than on v_{i+1}. In our example with three bidders, the cyclical crossing condition implies that signal s_1 has a higher influence on bidder 1's value than on bidder 2's value and a higher influence on bidder 2's value than on bidder 3's value: $v_{11} > v_{21} > v_{31}$. In the same fashion, signal s_2 has a higher impact on bidder 2's value than on bidder 3's value and a higher impact on bidder 3's value than on bidder 1's value: $v_{22} > v_{32} > v_{12}$. The same arguments imply that for signal s_3 $v_{33} > v_{13} > v_{23}$. We will illustrate the cyclical crossing condition with the following influence matrix, where arrows stands for the inequality >.

$$\left[\begin{array}{ccc} & \downarrow & \downarrow \\ v_{11} & v_{12} & v_{13} \\ \downarrow & & \downarrow \\ v_{21} & v_{22} & v_{23} \\ \downarrow & \downarrow & \\ v_{31} & v_{32} & v_{33} \\ & \downarrow & \downarrow \end{array}\right]$$

It is important to note that when there are two bidders both the average and cyclical crossing conditions reduce to the single crossing condition. However when there are three or more bidders and the average crossing condition is satisfied, bidders need not drop out in increasing order of their values in an English auction, even though the bidder with the highest value wins the auction. We will illustrate this phenomenon by using the example from Krishna (2002).

Suppose that signals are distributed on $[0,1]$ and that there are three bidders with values:

$$v_1 = s_1 + \frac{1}{3}s_2\,,\ v_2 = \frac{1}{3}s_1 + s_2\,,\ v_3 = \frac{1}{3}s_2 + s_3\,. \tag{23}$$

The average value is:

$$\overline{v} = \frac{4}{9}[s_1 + s_2 + s_3]\,. \tag{24}$$

First note that the influence of each bidder's signal on his value is higher than the influence on the average value since $v_{ii} = 1 > \overline{v}_i = \frac{4}{9}$. For signal s_1 we have the following inequalities:

$$v_{11} = 1 > \overline{v}_1 = \frac{4}{9} > v_{21} = \frac{1}{3} \text{ and } v_{11} = 1 > \overline{v}_1 = \frac{4}{9} > v_{31} = 0. \tag{25}$$

Thus the average crossing condition is satisfied for s_1. By using the same procedure it can be shown that the average crossing condition is satisfied for signals s_2 and s_3, as well.

Now suppose that the realized values of the signals are $(s_1, s_2, s_3) = \left(\frac{1}{100}, \frac{1}{10}, \frac{9}{10}\right)$. At the begining of the auction each bidder knows only his signal. We have shown that the equilibrium strategy for bidder 1 when all bidders are still active in an English auction is to drop out when the price posted by the auctioneer reaches $b_1^3(s_1, s_1, s_1) = v_1(s_1, s_1, s_1) = \frac{4}{3}s_1 = 0{,}0133$. Bidder 2 will drop out when the price reaches $b_2^3(s_2, s_2, s_2) = v_2(s_2, s_2, s_2) = \frac{4}{3}s_2 = 0{,}133$, while bidder 3 will drop out at the price $b_3^3(s_3, s_3, s_3) = v_3(s_3, s_3, s_3) = \frac{4}{3}s_3 = 1{,}2$. Therefore bidder 1 will drop out first, followed by bidder 2, while bidder 3 will win the auction.

By using the realized values of the signals we can conclude that $v_1 = 0{,}31$, $v_2 = 0{,}1033$, and $v_3 = 0{,}95$. Therefore bidder 3 has the highest *ex post* value and he wins the auction, which is efficient. On the other hand bidder 1 has a higher *ex post* value than bidder 2, but he quits the auction before bidder 2. This example shows us that when the average crossing condition is satisfied the allocation is efficient but it is possible that bidders do not drop out in order of increasing *ex post* values.

6. COMMON VALUE AUCTIONS

In the common value auctions all bidders have the same value for the object, but they only have an imprecise signal of that value. For example, if a bidder plans to resell the item, the resale price could be considered as a common value which is the same for all bidders but is unknown at the time of the auction. The winner's curse is particulary severe with common values. As we will see below, the bidding strategy is increasing in the bidder's signal and the bidder with the most optimistic signal wins the auction. But after the auction ends he learns the common value of the object, and he might discover that his signal was too optimistic and that the value of the object is lower than his bid. Hence rational bidders would shade their bids to avoid the winner's curse. Bazerman and Samuelson (1983) conducted an experiment in which they auctioned off jars filled with coins to MBA students at Boston University in a first-price auction. Each jar contained \$8, and this represents the common value of the object. They dicovered that the average bid was \$5.13 but the average winning bid was \$10.01, and winners suffered from the winner's curse.

We will consider the simplest case when the common value of the item (v) is the sum of two bidders' signals $v = s_1 + s_2$, where s_1 and s_2 are signals of bidders 1 and 2. For example, if the two bidders compete for a right to exploit oil in some area, s_1 could be an estimate of the amount of oil in one part of that area, and s_2 an estimate of the amount of oil in another part of that area. Suppose that the signals are independently and identically distributed according to uniform distribution at the interval $[0,1]$.

First-price auction

We will first derive equilibrium bidding strategies in a first-price auction by following Menezes and Monteiro (2004). Suppose that bidder 2 follows a strictly increasing strategy $b(s_2)$ and that bidder 1, who has a signal $S_1 = s$, bids as if his signal is $S_1 = z$, i.e. $b(z)$. Bidder 1 pays his bid if he wins, and his expected profit is:

$$\Pi(z) = \int_0^z (s + y - b(z))dy = sy + \frac{y^2}{2}\Big|_0^z - b(z)z = sz + \frac{z^2}{2} - b(z)z\ . \qquad (26)$$

By using the first order condition with respect to z, we obtain:

$$s + z - b(z) - b'(z)z = 0 . \tag{27}$$

In a symmetric equilibrium bidder 1 sets $z = s$ which implies:

$$2s - b(s) - b'(s)s = 0 , \tag{28}$$

$$(b(s)s)' = 2s . \tag{29}$$

Integrating both sides of this differential equation from 0 to s, we obtain:

$$s \cdot b(s) - 0 \cdot b(0) = \int_0^s 2y dy = s^2 . \tag{30}$$

By solving the last equation, we obtain the equilibrium bidding strategy $b(s) = s$. In other words, each bidder submits a bid equal to his signal and the bidder with the higher signal wins the auction. Since the common value of the object cannot be higher than his bid, the winning bidder does not suffer from the winners curse. We can conclude that bidding strategies in a setting with uniformly distributed signals are immune to the winner's curse. Unfortunately this result does not extend to some other signal distributions.

In the previous discussion we have analyzed the case with two bidders, but the severity of the winner's curse increases with the number of bidders, because having the highest signal among N+1 bidders is worse news than having the highest signal among N bidders. In fact Wolfstetter (1996) identifies two opposing effects when the number of bidders increases. The first effect is the *competitive effect*. When the number of bidders increases each bidder has an incentive to reduce bid shading, in order to increase his chances of winning the auction. The second effect is the *winner's curse effect*. It means that the severity of the winner's curse increases with the number of bidders and each bidder has an incentive to increase bid shading in order to avoid the winner's curse.

By following the logic described above Bulow and Klemperer (2002) point out the paradoxical result that increased competition can lead to lower prices due to the winner's curse. Contrary to common economic logic, preventing entry of some bidders or increasing the number of units sold reduces the winner's curse and might increase the seller's expected revenue.

Second-price auction

Now we will derive equilibrium bidding strategies in a second-price auction by following Menezes and Monteiro (2004). Suppose that bidder 2 follows a strictly increasing strategy $b(s_2)$ and that bidder 1, who has a signal $S_1 = s$, bids as if his signal is $S_1 = z$, i.e. $b(z)$. Bidder 1 pays the bid of the other bidder if he wins, and his expected profit is:

$$\Pi(z) = \int_0^z (s + y - b(y))dy \,, \tag{31}$$

By using Leibnitz's rule we have the following first-order condition:

$$s + z - b(z) = 0 \,. \tag{32}$$

In a symmetric equilibrium a bidder sets $z = s$, which implies that the equilibrium bidding strategy is $b(s) = 2s$.

Now let us consider the properties of this bidding strategy. The distribution of bidder 2's signal conditional on bidder 1 having the highest signal is:

$$F(y \,|\, s > y) = \frac{F(y)}{F(s)} = \frac{y}{s} \,, \tag{33}$$

where the last equality follows from the fact that signals are uniformly distributed. Thus the expected value of the object conditional on bidder 1 having the highest signal is:

$$E[v \,|\, s > y] = \int_0^s (s + y)dF(y \,|\, s > y) = \frac{1}{s}\int_0^s (s + y)dy = s + \frac{1}{s} \cdot \frac{s^2}{2} = \frac{3}{2}s \,. \tag{34}$$

The winning bidder bids higher than the the expected value of the object conditional on winning, but this does not mean that he suffers from the winner's curse because the winning bidder pays the second highest bid:

$$P = \int_0^s b(y)dF(y \mid s > y) = \frac{2}{s}\int_0^s ydy = \frac{2}{s}\cdot\frac{s^2}{2} = s \,, \tag{35}$$

where the second equality follows from the fact that bidder 2 bids according to $b(y) = 2y$ and the definition of the conditional distribution. Thus the equilibrium bidding strategy in a second-price auction is immune to the winner's curse as well, since the expected payment of the winning bidder is lower than the expected value conditional on winning.

Finally, note that the expected payment of the winning bidder in both auctions is the same and equal to s. This result is not a coincidence, since the revenue equivalence holds for the case of common values when bidders are risk neutral, symmetric, and signals are independently distributed. Hence the revenue equivalence extends to the case of interdependent values when signals are independent. Recall that we have assumed that signals were affiliated in the case of interdependent value, and that is why the revenue equivalence failed.

When signals are affiliated in the common value auction, the expected revenue ranking from the interdependent value case applies. The expected revenue from a second price auction is higher than the expected revenue from a first price auction because in a second price auction the winning bidder pays the second highest bid, whereas in a first-price auction he pays his bid and bids more cautiously. Thus the seller's expected revenue is higher in a second-price auction.

Finally, it is important to note that the issue of efficiency is unimportant in the common value case, since all bidders have the same value.

The wallet game

Klemperer (1998) uses the strategy in a second-price auction derived above in the so-called Wallet game. The two players have the amounts s_1 and s_2 of money in their wallets distributed according to $F_1(s_1)$ and $F_2(s_2)$. Each player knows the amount of money in his wallet, but does not know the amount of money in the other player's wallet. The auctioneer performs an English auction in which he raises the price gradually. When one player quits the winner obtains the combined contents of the two wallets and pays the auctioneer the price at which his runner-up dropped out. Thus the common value of the object is $v = s_1 + s_2$. Each bidder chooses a price at which to drop out conditional on his signal $b_1^2(s_1)$ and $b_2^2(s_2)$. The payoff of the winning bidder is $\Pi = s_1 + s_2 - \min(b_1(s_1), b_2(s_2))$. Since there are two bidders the English auction is strategically equivalent to a second-price auction and bidders use the same strategies as in a second-price auction. In other words a bidder stays in until the price reaches $b_i(s_i) = 2s_i$. Suppose that bidder 1 is the winner and that bidder 2 quits at $b_2(s_2)$. The value of the object for bidder 1 would be $v = s_1 + 0{,}5b_2(s_2)$ and he pays the auctioneer $b_2(s_2)$. Bidder 1 obtains a positive payoff if $s_1 + 0{,}5b_2(s_2) > b_2(s_2)$. By solving the last inequality we obtain that bidder 1's payoff is positive if $b_2(s_2) < 2s_1$. This justifies the use of the strategy $b_1(s_1) = 2s_1$ because when the price surpasses $b_1(s_1) = 2s_1$ the best strategy for bidder 1 is to quit the auction. The equilibrium described above is a *unique symmetric equilibrium*.

However there are many *asymmetric equilibria* in this auction when bidders use asymmetric bidding strategies. For example, bidder 1 might stay in until the price reaches $b_1(s_1) = 10s_1$ whereas bidder 2 quits at $b_2(s_2) = \frac{10}{9}s_2$. To verify that these strategies constitute an equilibrium, suppose that bidder 1 wins. Then the value of the object for bidder 1 would be $v = s_1 + \frac{9}{10}b_2(s_2)$. Bidder 1 obtains a positive payoff if $s_1 + \frac{9}{10}b_2(s_2) > b_2(s_2)$. By solving the last equation we can conclude that bidder 1 is pleased with winning if $b_2(s_2) < 10s_1$, and he prefers to quit when the price reaches $10s_1$. On the other hand, if bidder 2 wins, the value of the object would be $v = s_2 + \frac{1}{10}b_1(s_1)$. Bidder 2's payoff is positive if

$s_2 + \frac{1}{10} b_1(s_1) > b_1(s_1)$, which implies that bidder 2 is pleased with winning when $b_1(s_1) < \frac{10}{9} s_2$.

In this asymmetric equilibrium bidder 1 wins more often than in the symmetric equilibrium and when he wins he finds more money in bidder 2's wallet than in the symmetric equilibrium. Thus bidder 1's payoff is higher in the asymmetric equilibrium than in the symmetric equilibrium. On the other hand bidder 2 wins less often than in the symmetric equilibrium, and finds less money in bidder 1's wallet when he wins. Thus bidder 2's payoff is lower in the asymmetric than in the symmetric equilibrium. Finally, the seller is worse off in the asymmetric equilibrium. Klemperer (1998) reports the results of the simulation analysis in which bidder 1 wins 94% of the time and the seller's expected revenue is 20% lower than in the symmetric equilibrium.

Klemperer (1998) studies one more case of asymmetric equilibrium and discusses how the apparently small advantage of one player can be transformed into a large difference in bidding strategies in the English auction. Suppose that if bidder 1 wins he obtains a small bonus prize of 1, whereas bidder 2 does not obtain the bonus prize when he wins. This apparently small difference has a tremendous impact on equilibrium outcome, because bidder 1 always wins in equilibrium. In this case bidder 1 will bid more aggressively since he obtains an additional amount of 1 when he wins. In other words he bids as if his signal is $s_1 + 1$. But now the winner's curse is intensified for bidder 2 because when he wins he will find an amount of 1 less in his rival's wallet. Hence winning against a more aggressive bidder is worse news than winning against a less aggressive bidder. In order to avoid the winner's curse bidder 2 will bid less aggressively, as if his signal is $s_2 - 1$. But now bidder 1's winner's curse is reduced and he knows that he will find an additional amount of 1 in bidder 2's wallet when he wins, and he bids as if his signal is $s_1 + 2$. This magnifies the winner's curse further for bidder 2, and he will bid as if his signal is $s_2 - 2$, and so on. The small bonus prize of 1 translates into a large competitive advantage for bidder 1 in an English auction, and the seller's expected revenue could apparently be lower in this case than in symmetric equilibrium. In the above example bidder 1 always stays in.

Klemperer (1998) argues informally that in this case the seller should use a first-price auction, because bidders cannot condition their behaviour on other bidders' behaviour and cannot follow strange strategies such as staying in until the opponent quits. Therefore the asymmetric equilibrium of a first-price auction is close to symmetric equilibrium and a first-price auction yields higher expected revenue for the seller than an English auction. The case of asymmetric bidders is also examined by Bulow and Klemperer (2002), using a more formal approach.

Bikhchandani (1988) studies the case of repeated second-price common value auctions with two asymmetrically informed bidders. The first bidder is an ordinary bidder, whereas the second bidder is a strong bidder with probability δ. and an ordinary bidder with probability $1-\delta$. Bidder 2's type is his private information and the auction game is repeated finitely many times. Due to the effect explained earlier, it pays for bidder 2 to establish a reputation as an aggressive bidder in a second-price auction, even though he is an ordinary bidder. More precisely by bidding aggressively bidder 2 intensifies bidder 1's winner's curse, forcing him to bid more cautiously, which increases bidder 2's chances of winning. The winning bidder pays the second highest bid and since bidder 1 bids more cautiously, bidder 2 pays a lower price and the seller obtains a lower than expected revenue. Maintaing the reputation of an aggressive bidder is profitable in a repeated second-price auction. On the other hand it is costly for bidder 2 to maintain the reputation of an aggressive bidder in a first-price auction since he pays his bid. Thus a first-price auction is preferable over a second-price auction by the auctioneer in repeated auctions with asymmetrically informed bidders.

7. CONCLUDING REMARKS

Auction theory is an interesting and growing field of microeconomics. It shows us how abstract game theory models can be applied to practical auction design. Auctions are now widely used in some countries to sell antiquities, art objects, spectrum rights, pollution allowances, airport landing slots, oil drilling rights, etc. In recent years internet auctions have become very popular, particularly auctions on eBay. This phenomenon has motivated a substantial interest among theoretical researchers.

The assumption of interdependent values is more realistic than the assumption of private values, but auctions with interdependent values are more challenging from the modelling point of view. However in practical auction design affiliation is not such an important aspect as collusion, entry, or asymmetry between bidders.

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APPENDIX

A. Derivation of equilibrium bidding strategies in a first-price auction[4]

First note that the equilibrium bidding strategy must satisfy the boundary condition $b(0) = v(0,0) = 0$. Denote by:

$$\gamma(y) \equiv \frac{g(y\,|\,s)}{G(y\,|\,s)}. \tag{A1}$$

By substituting (A1) in (7), we obtain the following differential equation:

$$b'(s) + b(s)\cdot\gamma(s) = v(s,s)\cdot\gamma(s). \tag{A2}$$

Define the integrating factor as:

$$\psi(s) = \exp\left(-\int_s^{\omega}\gamma(u)du\right). \tag{A3}$$

By differentiating the expression $b(s)\cdot\psi(s)$, we obtain:

$$[b(s)\cdot\psi(s)]' = b'(s)\cdot\psi(s) + b(s)\cdot\psi'(s) = b'(s)\cdot\psi(s) + \gamma(s)\cdot\psi(s)\cdot b(s), \tag{A4}$$

where the second equality follows from the fact that $\psi'(s) = \gamma(s)\cdot\psi(s)$, where we have used Leibnitz's rule.

On the other hand, by multiplying (A2) with $\psi(s)$, we obtain:

$$b'(s)\cdot\psi(s) + b(s)\cdot\gamma(s)\cdot\psi(s) = v(s,s)\cdot\gamma(s)\cdot\psi(s). \tag{A5}$$

The left hand side of (A5) is equal to the right hand side of (A4), which implies:

$$[b(s)\cdot\psi(s)]' = v(s,s)\cdot\gamma(s)\cdot\psi(s). \tag{A6}$$

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4 This proof follows Menezes and Monteiro (2004).

By integrating the last equation between 0 and s, and by using the boundary condition $b(0)=v(0,0)=0$, we have that:

$$b(s)\cdot\psi(s)=\int_0^s \gamma(y)v(y,y)\cdot\psi(y)dy\ . \tag{A7}$$

By using the fact that $d\psi(s)=\gamma(s)\cdot\psi(s)$, it is obvious that:

$$b(s)=[\psi(s)]^{-1}\int_0^s v(y,y)\cdot d\psi(y)\ . \tag{A8}$$

Substituting $\psi(s)=\exp\left(-\int_s^\omega \gamma(u)du\right)$ and $\psi(y)=\exp\left(-\int_y^\omega \gamma(u)du\right)$ in the last expression, we obtain that:

$$b^*(s)=\int_0^s v(y,y)dL(y\,|\,s)\ , \tag{A9}$$

where

$$L(y\,|\,s)=\exp\left(-\int_y^s \frac{g(t\,|\,t)}{G(t\,|\,t)}dt\right). \tag{A10}$$

Finally, note that $L(\cdot\,|\,s)$ can be considered as a distribution function with the support $[0,s]$. It can be shown that $L(0\,|\,s)=0$, $L(s\,|\,s)=1$ and that $L(\cdot\,|\,s)$ is non-decreasing, which implies that $L(\cdot\,|\,s)$ is a distribution function. ▪

B. Proof that equilibrium bidding strategies in first-price and second-price auctions are immune to the winner's curse[5]

Denote by $K(y\,|\,s)$ conditional distribution of the second highest signal Y_1 conditional on $S_1=s$ and $Y_1<s$. By construction,

$$K(y\,|\,s)=\frac{G(y\,|\,s)}{G(s\,|\,s)}\ . \tag{B1}$$

[5] This proof is based on Krishna (2002).

Since values are affiliated, it follows that for any two values $t < s$, the conditional distribution $G(\cdot \mid s)$ dominates $G(\cdot \mid t)$ in terms of the reverse hazard rate:

$$\frac{g(t \mid t)}{G(t \mid t)} \leq \frac{g(t \mid s)}{G(t \mid s)}. \tag{B2}$$

We can integrate the last expression from s to y, for $y < s$, which gives us:

$$-\int_y^s \frac{g(t \mid t)}{G(t \mid t)} dt \geq -\int_y^s \frac{g(t \mid s)}{G(t \mid s)} dt = -\int_y^s \frac{d}{dt} (\ln G(t \mid s)) dt = \ln G(y \mid s) - \ln G(s \mid s) = \ln\left(\frac{G(y \mid s)}{G(s \mid s)}\right). \tag{B3}$$

Applying the exponential function to both side of the last inequality, we obtain:

$$L(y \mid s) = \exp\left(-\int_y^s \frac{g(t \mid t)}{G(t \mid t)} dt\right) \geq K(y \mid s) = \frac{G(y \mid s)}{G(s \mid s)}. \tag{B4}$$

Therefore the conditional distribution $K(y \mid s)$ dominates the conditional distribution $L(y \mid s)$ in terms of first order stochastic dominance.

We know that equilibrium bidding strategy in a first-price auction is:

$$b^*(s) = \int_0^s v(y, y) dL(y \mid s) \leq \int_0^s v(y, y) dK(y \mid s) \leq \int_0^s v(s, y) dK(y \mid s) = E[V_1 \mid S_1 = s, Y_1 < s]. \tag{B5}$$

The first inequality follows from the fact that $K(y \mid s)$ dominates $L(y \mid s)$ according to first order stochastic dominance, second from the fact that the value function is strictly increasing and that $y < s$, and the final equality follows from the fact that $K(y \mid s)$ represents the conditional distribution of the second highest value Y_1 conditional on $S_1 = s$ and $Y_1 < s$. Hence from (B5) it follows that equilibrium bid in a first-price auction is lower than the expected value conditional on winning and the bidding strategy is immune to the winner's curse.

The expected payment of the winning bidder in a second-price auction is $E[v(y,y)\,|\,S_1=s,Y_1<s]$, since the winning bidder pays the second highest bid which is equal to $v(y,y)$. Therefore, by using the same arguments given above, we have that:

$$E[v(y,y)\,|\,S_1=s,Y_1<s]=\int_0^s v(y,y)dK(y\,|\,s)\le\int_0^s v(s,y)dK(y\,|\,s)=E[V_1\,|\,S_1=s,Y_1<s]\,. \quad \text{(B6)}$$

The last result shows us that the expected payment of the winning bidder in a second-price auction is lower than the expected value conditional on winning, which implies that bidding strategies in a second-price auction are immune to the winner's curse as well.

One more result is worth noting. From (B5) and (B6) it follows that:

$$b^*(s)=\int_0^s v(y,y)dL(y\,|\,s)\le\int_0^s v(y,y)dK(y\,|\,s)=E[v(y,y)\,|\,S_1=s,Y_1<s]\,. \quad \text{(B7)}$$

The last results shows us that the expected payment of the winning bidder is at least as high in a second-price auction as in a first-price auction, and we have formaly proved the expected revenue ranking of these two auctions. ■

C. Expected revenue ranking of the war of attrition and a second-price auction[6]

We will first derive bidding strategy in the war of attrition. Consider bidder 1 who has a signal $S_1=s$ and submits a bid b, whereas other bidders follow the increasing strategy $b(\cdot)$. Bidder 1's expected payoff is:

$$\Pi(b,s)=\int_0^{b^{-1}(b)}(v(s,y)-b(y))g(y\,|\,s)dy-[1-G(b^{-1}(b)\,|\,s)]\cdot b\,, \quad \text{(C1)}$$

where the first term represents the profit of the bidder if he wins and pays the second highest bid, whereas the second term represents the case when he loses the auction and pays his bid.

6 This proof is based on Krishna and Morgan (1997).

Maximizing the profit function with respect to *b*, and using Leibnitz's rule, we obtain the following first order condition:

$$v(s,b^{-1}(b))g(b^{-1}(b)\,|\,s)\frac{1}{b'(b^{-1}(b))}-b\cdot g(b^{-1}(b)\,|\,s)\frac{1}{b'(b^{-1}(b))}+b\cdot g(b^{-1}(b)\,|\,s)\frac{1}{b'(b^{-1}(b))}-[1-G(b^{-1}(b)\,|\,s)]=0 \qquad \text{(C2)}$$

$$v(s,b^{-1}(b))g(b^{-1}(b)\,|\,s)\frac{1}{b'(b^{-1}(b))}-[1-G(b^{-1}(b)\,|\,s)]=0\,. \qquad \text{(C3)}$$

In a symmetric equilibrium, bidder 1 bids according to $b=b(s)$, and we obtain the following differential equation:

$$b'(s)=v(s,s)\frac{g(s\,|\,s)}{1-G(s\,|\,s)}=v(s,s)\cdot\lambda(s\,|\,s)\,, \qquad \text{(C4)}$$

where $\lambda(s\,|\,s)$ represents the hazard rate. Integrating the last expression between 0 and *s*, we obtain:

$$b(s)=\int_0^s v(y,y)\lambda(y\,|\,y)dy\,. \qquad \text{(C5)}$$

The expected payment of a bidder in the war of attrition is:

$$P^w(s)=\int_0^s b(y)g(y\,|\,s)dy+[1-G(s\,|\,s)]\cdot b(s)\,. \qquad \text{(C6)}$$

Integrating the integral by parts, we obtain:

$$P^w(s)=b(s)G(s\,|\,s)-\int_0^s b'(y)G(y\,|\,s)dy+[1-G(s\,|\,s)]\cdot b(s)=b(s)-\int_0^s b'(y)G(y\,|\,s)dy\,. \qquad \text{(C7)}$$

By substituting (C5) in the last expression, we obtain:

$$P^w(s)=\int_0^s v(y,y)\lambda(y\,|\,y)dy-\int_0^s v(y,y)\lambda(y\,|\,y)G(y\,|\,s)dy=\int_0^s v(y,y)\lambda(y\,|\,y)g(y\,|\,s)\left[\frac{1-G(y\,|\,s)}{g(y\,|\,s)}\right]dy \qquad \text{(C8)}$$

By using the definition of the hazard rate, we obtain:

$$P^{w}(s)=\int_{0}^{s}v(y,y)g(y\,|\,s)\left[\frac{\lambda(y\,|\,y)}{\lambda(y\,|\,s)}\right]dy\,. \tag{C9}$$

On the other hand, we have proved that a bidder with a signal s will bid $b(s)=v(s,s)$ in a second-price auction. His expected payment is:

$$P^{II}(s)=\int_{0}^{s}v(y,y)g(y\,|\,s)dy\,. \tag{C10}$$

It can be shown that due to the affiliation for any $s>y$ the hazard rate is decreasing $\lambda(y\,|\,s)\leq\lambda(y\,|\,y)$, which implies that the expected revenue in the war of attrition is higher than the expected revenue in a second-price auction. ▪

D. Expected revenue ranking of an all-pay auction and a first-price auction[7]

We will first derive bidding strategy in the all-pay auction. Consider bidder 1 who has a signal $S_1=s$ and submits a bid b, whereas other bidders follow the increasing strategy $b(\cdot)$. Bidder 1's expected payoff is:

$$\Pi(b,s)=\int_{0}^{b^{-1}(b)}v(s,y)g(y\,|\,s)dy-b\,, \tag{D1}$$

where the first term represents his value of the object if he wins and the second term represents the payment that he has to make regardless of whether he wins or not.

Maximizing the profit function with respect to b, and using Leibnitz's rule, we obtain the following first order condition:

$$v(s,b^{-1}(b))g(b^{-1}(b)\,|\,s)\frac{1}{b'(b^{-1}(b))}-1=0\,. \tag{D2}$$

In a symmetric equilibrium, bidder 1 bids according to $b=b(s)$, and we obtain the following differential equation:

7 This proof is based on Krishna and Morgan (1997).

$$b'(s) = v(s,s)g(s \mid s)\,. \tag{D3}$$

Integrating the last expression between 0 and s, we obtain the bidding strategy in the all-pay auction:

$$b(s) = \int_0^s v(t,t)g(t \mid t)dt\,. \tag{D4}$$

The equilibrium bidding strategy in a first-price auction can be written as:

$$b^*(s) = \int_0^s v(y,y)dL(y \mid s)\,, \tag{D5}$$

where

$$L(y \mid s) = \exp\left(-\int_y^s \frac{g(t \mid t)}{G(t \mid t)}dt\right). \tag{D6}$$

The expected payment of a bidder in a first-price auction is:

$$P^I(s) = G(s \mid s)\cdot b^*(s) = G(s \mid s)\cdot \int_0^s v(y,y)dL(y \mid s)\,. \tag{D7}$$

Differentiating (D6), we obtain:

$$P^I(s) = \int_0^s v(y,y)g(y \mid y)\left[\frac{G(s \mid s)}{G(y \mid y)}\right]\exp\left(-\int_y^s \frac{g(t \mid t)}{G(t \mid t)}dt\right)dy\,. \tag{D8}$$

Due to the affiliation, for any $t > y$:

$$\frac{g(t \mid t)}{G(t \mid t)} \geq \frac{g(t \mid y)}{G(t \mid y)}\,. \tag{D9}$$

Integrating the last inequality from y to s and multiplying by (-1), we obtain:

$$-\int_y^s \frac{g(t\mid t)}{G(t\mid t)}dt \le -\int_y^s \frac{g(t\mid y)}{G(t\mid y)}dt = \ln G(y\mid y) - \ln G(s\mid y) \le \ln G(y\mid y) - \ln G(s\mid s)\ , \tag{D10}$$

where the last inequality follows from the fact that $G(s\mid\cdot)$ is non-increasing in the second argument due to the affiliation. Taking the exponent of both sides of (D10), we have that:

$$\exp\left(-\int_y^s \frac{g(t\mid t)}{G(t\mid t)}dt\right) \le \frac{G(y\mid y)}{G(s\mid s)}\ . \tag{D11}$$

$$\left[\frac{G(s\mid s)}{G(y\mid y)}\right]\exp\left(-\int_y^s \frac{g(t\mid t)}{G(t\mid t)}dt\right) \le 1\ . \tag{D12}$$

On the other hand, the expected payment in the all-pay auction is:

$$P^A(s) = \int_0^s v(y,y)g(y,y)dy\ . \tag{D13}$$

By using (D8) and (D12), we obtain that:

$$P^A(s) = \int_0^s v(y,y)g(y,y)dy \ge P^I(s) = \int_0^s v(y,y)g(y\mid y)\left[\frac{G(s\mid s)}{G(y\mid y)}\right]\exp\left(-\int_y^s \frac{g(t\mid t)}{G(t\mid t)}dt\right)dy\ . \tag{D14}$$

Hence, the expected payment in the all-pay auction is higher than in a first-price auction. ▪

E. Comparison of different levels of reserve prices and entry fees[8]

Define the screening level as the lowest possible signal that would make bidder 1 participate in the auction, when he faces the pair of a reserve price r and an entry fee e, has a signal $S_1 = s$, and the second highest signal is Y_1:

$$s^* = \inf\{s\mid E[(V_1 - r)1_{\{Y_1<s\}}\mid S_1 = s] \ge e\}\ . \tag{E1}$$

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8 The proof follows Milgrom and Weber (1982).

We call a pair (r, e) a *regular pair* if all bidders with a screening level higher than s^* participate in the auction. Consider another regular pair $(\bar{r}, \bar{e})$ that lead to the same screening level, such that $r < \bar{r}$ and $\bar{e} < e$.

Denote by $P(z, s)$ the expected payment of a bidder 1 in the (r, e) auction when his signal is $S_1 = s$ but he bids as if his signal is z and other bidders follow their equilibrium strategies. Define $\bar{P}(z, s)$ in the same way for the $(\bar{r}, \bar{e})$ auction.

In a first price auction the expected payments of a bidder are $P(z, s) = b(s) \cdot G(z \mid s) + e$ and $\bar{P}(z, s) = \bar{b}(s) \cdot G(z \mid s) + \bar{e}$, where $G(\cdot \mid s)$ is the distribution function of the second highest signal conditional on $S_1 = s$. We have seen in appendix A that equilibrium bidding strategies in a first-price auction are obtained after solving a differential equation. Note that $b(s)$ and $\bar{b}(s)$ are obtained as a solution to the same differential equation and a bidder whose signal is equal to the screening level submits a bid equal to the reserve price $b(s^*) = r < \bar{r} = \bar{b}(s^*)$, which implies that $b(s)$ and $\bar{b}(s)$ cannot cross and $b(s) < \bar{b}(s)$. By finding the partial derivative of the expected payment function with respect to z at a point $s = z$, we obtain the following equation:

$$\frac{\partial P(z,s)}{\partial s} - \frac{\partial \bar{P}(z,s)}{\partial s} = [b(s) - \bar{b}(s)] \cdot \frac{\partial}{\partial s}\Big|_{s=z} G(z \mid s) \geq 0, \tag{E2}$$

where the last inequality follows from the fact that $b(s) < \bar{b}(s)$ and from the fact that the distribution function $G(z \mid \cdot)$ is non-increasing, due to affiliation. By using the boundary condition, $b(s^*) < \bar{b}(s^*)$ and by using (E2), it follows that $P(z, s) \geq \bar{P}(z, s)$.

In a second-price auction a bidder's expected payment is only affected when he pays the reserve price, and by mimicking the arguments used in the preceding proof we obtain that:

$$\frac{\partial P(z,s)}{\partial s} - \frac{\partial \bar{P}(z,s)}{\partial s} = [r - \bar{r}] \cdot \frac{\partial}{\partial s}\Big|_{s=z} G(z \mid s) \geq 0, \tag{E3}$$

and $P(z,s) \geq \overline{P}(z,s)$.

Since the expected payment of a bidder in a first-price and a second-price auction is higher in the (r,e) auction than in the $(\overline{r},\overline{e})$ auction, it follows that the seller's expected revenue is higher in the (r,e) auction. ▪

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