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# A NOTE ON THE ARROW'S IMPOSSIBILITY THEOREM**

**ABSTRACT:** *This paper generalizes Arrow's impossibility theorem (Arrow1950) in two directions. First we allow agents to have incomplete preferences, and the admissible domain of preferences can be a proper subset of the full domain.*

**KEY WORDS:** *Arrow Impossibility Theorem, Social Welfare Function, Incomplete Preferences*

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## 1. INTRODUCTION

Arrow's impossibility theorem (Arrow1950) is one of the most fundamental results in the theory of collective choice. It says that there is no `satisfactory' method (to be made precise below) to aggregate individual preferences into a consistent social preference. Given its importance, there is a large literature that studies its various extensions. However, only a few studies focus on the completeness axiom; for example, Pini et al.(2009). To continue this line of inquiry this paper studies Arrow's impossibility theorem (Arrow 1950) while allowing agents to have incomplete preferences. Recently, this journal also published related papers dealing with logical relationships between dictatorship, liberalism, and the Pareto rule (see Boricic 2009), as well as with the forms of impossibility theorem in multiple von Wright's preference logic (see Boricic 2014)

Before proceeding, let us note that there are several reasons why one would like to model agents with incomplete preference relations. Firstly, it is not evident that completeness of preference relation is a fundamental property of rationality, the way transitivity is. This view has been advanced in the literature (see, for example, Aumann1962). Secondly, having a complete consistent ranking of alternatives is cognitively very costly, especially when the number of alternatives is large. In such circumstances it would be convenient for an agent to identify his best two or three alternatives and rank them consistently, possibly leaving the rest unranked. Thirdly, there are instances when a single agent represents or is composed of multiple agents; in such cases requiring complete preferences is very demanding. For instance, two countries are voting on a set of issues: each country might be sure of its best alternatives but could be indecisive about the other alternatives due to lack of unanimity on the other issues.

Two classes of incomplete preferences have been studied in the literature. Reffgen (2011) studies Gibbard-Satterthwaite's theorem (Gibbard 1973, Satterthwaite 1975) under a class of incomplete preference where agents identify their top $k$ among $n$ alternatives and have complete preferences among them but are allowed to have incompleteness in the bottom $n$-$k$ alternatives. Pini et.al (2009) studies Arrow's theorem under a class of preferences where agents identify their best and worst alternative among $n$ alternatives, but are allowed to have incompleteness in-between. This paper studies Arrow's theorem, as in Pini et al. (2009), under the class of incomplete preference identified by Reffgen (2011). Thus this paper complements both of the above.

Incomplete preferences have also been studied in other areas of economics such as choice theory (Eliaz and Ok 2006, Mandler2005) and expected utility theory (Dubra et. al. 2004). In the context of strategic voting, Reffgen (2011) proves the Gibbard-Satterthwaite theorem when the admissible domain is a proper subset of the full domain.

## 2. SET UP

Let $X = \{x,y,z,..\}$ be the finite set of alternatives, *s.t.* $|X| \geq 3$, and $R$ be the set of all partial orders (reflexive and transitive) over $X$. Let $N$ denote the set of individuals, $|N| \geq 2$. Every individual $i \in N$ is said to have a reflexive and transitive binary preference relation over $X$ denoted by $R_i \in \mathcal{R}$. We denote $P_i$ as the asymmetric part and $I_i$ as the symmetric part of $R_i$. A preference profile is a tuple $(R_1,....,R_n)$ where $R_i \in \mathcal{R}$. Denote $\mathcal{R}^N$ as the set of all possible profiles. In this paper we restrict agents to having strict preferences; i.e., an agent cannot be indifferent between two distinct alternatives. Formally, if $xR_iy$ and $yR_ix$, then $x=y$. We make this assumption only for the sake of exposition. Our main theorem is true even without this assumption.

In what follows we identify a class of incomplete preferences. To do this we introduce the notion of *Top-k* preference and connectedness. This class of incomplete preferences was introduced in Reffgen (2011) in the context of social choice functions. The basic idea is that agents are assumed to rank the *Top-k* alternatives but are allowed to have incomplete preferences over the bottom *n-k* alternatives. The notion of connectedness captures the idea of conflict of interest; i.e., the possibility that any two agents can disagree over the ranking of alternatives, say $x$ and $y$.

Definition 2.1 (Not Compared). We say an agent $i \in N$ is *not able to compare two distinct alternatives x and y*, if, according to his preference relation, he is unable to rank $x$ and $y$, i.e, neither $xP_iy$ nor $yP_ix$. We denote this by $xN_iy$

Definition 2.2 (Top $k$ Preference). Let $k$ be a positive integer, $2 \leq k \leq M$. A transitive and asymmetric binary relation $P \in \mathcal{R}$ on a set $A$ is a top-$k$ preference if there exist $k$ alternatives $r_1(P), \ldots, r_k(P) \in X$ such that (1) $r_j(P)Pa$ for every $j = 1, \ldots, k$ and $a \in X$ $r_1(P), \ldots, r_k(P)$, and (2) if $k \leq 1$, then $r_j(P)\, P\, r_{j+1}(P)$ for every $j = 1, \ldots, k\text{-}1$. Here, $r_i(P)$ denotes the $i^{th}$ ranked alternative under $P$

Let $\mathbb{D}_k \subseteq \mathcal{R}$ denotes a Top-$k$ domain.

Definition 2.3 (Connectedness). Two distinct alternatives $x, x^{'} \in$ X are *connected* in $\mathbb{D}_k$ if there exist $P, P^{'} \in \mathbb{D}_k$ such that $r_1(P) = r_2(P^{'}) = x$ and $r_1(P^{'}) = r_2(P) = x^{'}$. If $x$ and $x^{'}$ are connected, we write $x\,C\,x^{'}$.

Definition 2.4 (Free Triples). Alternatives $x,y,z \in X$ are said to be *free triples* if $x\,C\,y$, $y\,C\,z$ *and* $x\,C\,z$.

Arrow (1950) assumes a full domain of preferences: we relax this assumption here and generalize the notion of saturating domain introduced in Kalai et al. (1979) for incomplete preferences.

Definition 2.5 (Saturating Top $k$ Domain). $\Gamma_k \subseteq \mathbb{D}_k$ is said to be a *Saturating Top k domain* if there exists (i) At least two distinct connected pairs and (ii) for any two connected pairs $A,B$ contained in $X$, there exists a finite sequence of pairs $A_1,A_2,\ldots,A_r$, s.t. $A_1=A$ and $A_r=B$, s.t. $A_j,A_{j+1}$ form a free triple[1].

Note that $\Gamma_k \subseteq \Gamma_{k-1}$ , for every $k \in \{3,4,\ldots,n\}$.

The problem of social choice is to design satisfactory decision procedures which aggregate individual preferences into a social ordering over $X$. We define a social welfare function as a mapping from a set of preference profiles into a set of partial orders over $X$ i.e. $\mathcal{R}$. Let us denote $R \in \mathcal{R}$ as the social welfare ordering corresponding to $(R_1,\ldots,R_n)$. We denote this mapping by $F$. For a profile $(R_1,\ldots,R_n) \in \Gamma_k^N$, we denote $F(R_1,\ldots,R_n) \in \mathcal{R}$ as $R$. Therefore $xF(R_1,\ldots,R_n)y$ is written as $xRy$.

Definition 2.6 (Independence of Irrelevant alternatives ($I^*$)). For any pair of alternatives $x,y \in X$ and any pair of preference profiles $(R_1,\ldots,R_n),(R'_1,\ldots,R'_n) \in \Gamma_k^N$, if the preference between $x$ and $y$ stay the same then the society's preference,

[1] This implicitly means that $Aj \cup A_{j+1}$ have three alternatives

given by the social welfare function, between $x$ and $y$ does not change. More formally, a Social welfare function satisfy condition ($I^*$) if for any pair of preference profiles $(R_1,\ldots,R_n)$, $(R'_1,\ldots,R'_n) \in \Gamma_k^N$ and for any pair of alternatives $x,y \in X$ if $[\{(xP_iy \Leftrightarrow (xP'_iy)\} \wedge \{(yP_ix \Leftrightarrow (y P'_i x)\} \wedge \{(yN_ix \Leftrightarrow (yN'_i x)\}]$ then $[\{(xRy \Leftrightarrow (xR'y)\} \wedge \{(yRx \Leftrightarrow (yR'x)\}]$

Note that $(I^*)$ generalizes Arrow's Independence of irrelevant alternatives $(I)$ (Arrow 1950).It is easy to see that in the world of complete preferences, $(I^*)$ is equivalent to $I$.

Definition 2.7 (Pareto ($P^*$)). For any pair of alternative $x,y \in X$ whenever everyone in the society strictly prefers $x$ over $y$ then the society should also strictly prefer $x$ over $y$.

Definition 2.8 (Dictatorial ($D^*$) on $X' \subseteq X$). A social welfare function is *dictatorial* on the set $X' \subseteq X$ if there exists an agent $i \in N$ s.t. for any pair of alternatives in $X'$ whenever agent $i$ prefers $x$ over $y$ then the society also strictly prefers $x$ over $y$.

Definition 2.9 (Completeness Axiom). For any $x,y \in X$, if $(\text{for every } i \in N)(xP_iy \vee yP_ix)$ then $(xRy \vee yRx)$

## 3. MAIN RESULT

The main question addressed in the paper is whether Arrow's theorem still holds under this class of incomplete preferences. In other words, whether conditions $I^*$ and $P^*$ are mutually consistent with non-dictatorship on $\Gamma_k$. We show that the negative result in Arrow (1950) holds under our set-up. In what follows we assume the completeness axiom, unless otherwise stated.

Theorem 1 Any social welfare function $F: \Gamma_k^N \mapsto \mathcal{R}$, with $|X| \geq 3$ and $|N| \geq 2$, satisfying IIA ($I^*$) and Pareto $(P^*)$ is Dictatorial $(D^*)$ on $X$ for any $k \in \{2,3,\ldots,n\}$.

In what follows we prove the above theorem for the case of $k = 2$, in view of the remark that the case of $k$ greater than 2 follows as corollary. Before proceeding to the proof, we need to introduce some definitions.

Definition 3.1 (Almost Decisiveness). A group $V \subseteq N$ is *almost decisive* over an ordered pair $(x,y)$ if for any preference profile, everyone in the group prefers $x$ over $y$ and rest of the society prefers $y$ over $x$, then the social preference is $x$ over $y$. Formally, for any profile $(R_1,....,R_n) \in \Gamma^N$ if for every $i \in V$, $xP_iy$ and for every $i \in N \setminus V$, $yP_ix$ then $xPy$. We denote this fact by $\overline{D}_{V(x,y)}$.

Definition 3.2 (Decisiveness). A group $V \subseteq N$ is *decisive* over an ordered pair $(x,y)$ if for any preference profile, everyone in the group prefers $x$ over $y$, then the social preference is $x$ over $y$. Formally for any profile $(R_1,....,R_n) \in \Gamma^N$ if for every $i \in V$, $(xP_iy)$ then $(xPy)$. Denote this fact by $D_{V(x,y)}$

Definition 3.3 (Decisive Set). A set $V \subseteq N$ is *decisive* set if it is Decisive over all ordered pairs $(a,b) \in \{x,y,z\}$

Definition 3.4 (Minimal Decisive Set). A set $V \subseteq N$ is *minimal decisive* set if there does not exists a $V'$ s.t. $V' \subseteq V$ and $V'$ is a decisive set.

Lemma 1(Field Expansion Lemma). A group $V$ is Almost Decisive over $(x,y)$ then it is Decisive over all ordered pairs from $\{x,y,z\}$, i.e. $V$ is a decisive set

Proof. We first prove two steps, step 1 and step 2. These steps would be very useful to prove the Field Expansion Lemma.

Step 1: $\overline{D}_{V(x,y)} \Longrightarrow D_{V(x,z)}$

Step 1 says that if a group is almost decisive over an ordered pair $(x,y)$ then it is decisive over every ordered pair $(x,z)$

It would be convenient to present the preference profile using a table. Each column of the table represents the preferences of an individual or group of individuals. Top ranking alternatives come first, followed by second, and so on. For example, if an agent has preferences $xP_iyP_iz$ then in the $j^{th}$ column $x$ comes first, $y$ comes second, and $z$ comes third, and so on.

Consider the following preference profiles.

| **V** | **N\V** | **V** | **N\V** | **V** | **N\V** | **V** | **N\V** | V | N\V |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | $y$ | $x$ | $y$ | $x$ | $y$ | $x$ | $y$ | $x$ | . |
| $y$ | . | $y$ | $z$ | $y$ | $x$ | $y$ | $w$ | $y$ | . |
| . | . | . | . | . | . | . | . | . | . |
| (i) | | (ii) | | (iii) | | (iv) | | (v) | |

For Profiles (*i*) to (*iv*), *xPy*, since group *V* is almost decisive over (*x,y*) i.e., $\overline{D}_{V(x,y)}$. Also *yPz* by (*P**), hence, using transitivity of *P* we get *xPz* at profiles *(i)* to *(iv)*. By (*I**) we get *xPz* at profile (*v*). To see this, notice that for profiles *(i)* to *(iv)* the social ranking between *x* and *z* does not depend on the ranking of *x* and *z* for agents in *N \ V*.

Step 2: $D_{V(x,z)} \Longrightarrow \overline{D}_{V(y,z)}$.

Step 2 says that if a group is decisive over an ordered pair (*x,z*) then it is almost decisive over every ordered pair (*y,z*)

Consider the following profile.

| **V** | **N\V** |
|---|---|
| $y$ | $z$ |
| $x$ | $y$ |
| $z$ | . |

At this profile, *xPz*, since $\overline{D}_{V(x,z)}$ and *yPx* by *(P*)*, hence, by transitivity of *P* we have *yPz*. *V* is almost decisive over (*y,z*).

Now using step 1 and step 2 we show that if a group is almost decisive over an ordered pair (*x,y*) then it is decisive over all ordered pairs from *{x,y,z}*.The proof follows a series of steps.

Let us assume that a group *V* is almost decisive over (*x,y*); i.e., $\overline{D}_{V(x,y)}$, then using step 1 group *V* is decisive over (*x,z*); i.e.,$D_{V(x,z)}$. Now since $D_{V(x,z)}$ is true, then by step 2 group *V* is almost decisive over (*y,z*). Now given $\overline{D}_{V(y,z)}$, then, using step 1, again $D_{V(y,x)}$. Now, again using step 1, $\overline{D}_{V(y,x)}$ implies, $D_{V(y,z)}$ and using $\overline{D}_{V(x,z)}$ implies $D_{V(x,y)}$. Using symmetric argument $D_{V(z,x)}$ and $D_{V(z,y)}$ is true.

Lemma 2(Group Contraction Lemma). Let $V$ be a minimal decisive set. Then $|V|= 1$

Proof. Let $|V|\geq 2$ and let be a $\{V_1, V_2\}$ partition $V$. Consider the following profile.

| $V_1$ | $V_1$ | $N\backslash V_1$ |
|---|---|---|
| $x$ | $y$ | $z$ |
| $y$ | $z$ | $x$ |
| $z$ | $x$ | $y$ |

At this profile, $yPz$ by decisiveness of $V$. By completeness axiom, $R$ is complete over $x$ and $y$, i.e., $xRy \vee yPx$. If $yPx$, then $V_2$ is almost decisive over $(x,y)$ and by Lemma 1 above $V_2$ is a decisive set. If $xRy$, then by transitivity we have $xPz$, but this means that $V_1$ is almost decisive over $(x,z)$ and by Lemma 1 above $V_1$ is a decisive set. Therefore we get contradiction. In this way we can partition any decisive set until we arrive at a set such that we cannot further partition it. This minimal decisive set is of cardinality one.

The use of Field expansion and Group Contraction Lemma is one of the classical ways to prove Arrow's Impossibility Theorem; see, for example, Sen 2011, 1970. Now we are ready to prove the following statement.

Lemma 3. Let $\{a,b,c\}$ be a free triple then any such social welfare function satisfying IIA ($I^*$) and Pareto ($P^*$) is Dictatorial ($D^*$) on $\{a,b,c\}$.

Proof. By Pareto axiom ($P^*$) there exists a decisive set, therefore a minimal decisive set always exists. By Lemma 2 the minimal decisive set is singleton.

Lemma 3 is related to a free triple $\{a,b,c\}$. Next we extend this result to the entire set of alternatives $X$. To do this we use the concept of the saturating domain. The idea is that the saturating domain contains `enough' free triples, which allow us to extend Lemma 3 to $X$. We restate Theorem 1 and provide a proof for the same.

Theorem (Theorem 1) Any social welfare function $F: \Gamma_k^N \longmapsto \mathcal{R}$, with $|X| \geq 3$ and $|N| \geq 2$, satisfying IIA ($I^*$) and Pareto ($P^*$) is Dictatorial ($D^*$) on X for any $k \in \{2,3,\ldots,n\}$.

Proof. Since we have Top-2 saturating domain, there are at least two distinct connected pairs, say $A$ and $B$. Now by the definition of Top-2 saturating domain we can find a sequence $A_1, A_2, \ldots, A_r$, $s.t. A_1=A$ and $A_r=B$, $s.t.$ pair $A_j$ and $A_{j+1}$ form a free triple. Now by Lemma 3 every free triple has a dictator. In particular, every free triple in this sequence has a dictator; i.e., every $A_j \cup A_{j+1}$ has a dictator. We argue that this dictator is the common dictator. Let $i \in N$ be a dictator over $A_1 \cup A_2$ and $j \in N$ be a dictator over $A_2 \cup A_3$ and let $i \neq j$. This means that both $i$ and $j$ are dictators over $A_2$, which is a contradiction. Therefore $i=j$. This argument extends to all the free triples in the sequence and hence there exists an agent $i \in N$, who is the dictator over the entire set $X$.

Note that if we assume agents to have weak orders (complete, reflexive, and transitive) then $I$ implies $I^*$, and Arrow's theorem (Arrow1950) becomes a corollary to our result, bearing in mind that conditions ($P^*$) and ($D^*$) are defined independently of the domain.

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