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## SEQUENTIAL AUCTIONS AND PRICE ANOMALIES

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**ABSTRACT:** *In sequential auctions objects are sold one by one in separate auctions. These sequential auctions might be organised as sequential first-price, second-price, or English auctions. We will derive equilibrium bidding strategies for these auctions. Theoretical models suggest that prices in sequential auctions with private values or with randomly assigned heterogeneous objects should have no trend. However, empirical research contradicts*

*this result and prices exhibit a declining or increasing trend, which is called declining and increasing price anomaly. We will present a review of these empirical results, as well as different theoretical explanations for these anomalies.*

**KEY WORDS:** *Sequential first-price auctions; Sequential second-price auctions; Martingale property of prices; Declining price anomaly; Increasing price anomaly.*

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## **1. INTRODUCTION**

In this paper we will present one special form of multi-unit auction, when objects are sold one by one in sequential auctions<sup>1</sup>. In contrast to multi-unit auctions where all objects are sold at the same time, sequential auctions last longer and bidders can obtain information concerning the valuations of previous round-winners. As in single-unit auctions, bidders might have private or interdependent values. Sequential auctions might be organised as sequential first-price, second-price, or English auctions. Equilibrium strategies in these auctions could be derived recursively, starting from the last period. In order to obtain the tractable solution, the most common assumption is that bidders have single-unit demand, which means that the winner of one auction will not participate in other auctions. Milgrom and Weber (2000) derive equilibrium bidding strategies in sequential first-price auctions, second-price auctions, and sequential English auctions with private and interdependent values. When bidders demand more than one unit, equilibrium strategies are more complicated, and in this paper we will extend our analysis only to the case of two-unit demand.

We will see that bidding strategy is strictly increasing in both first-price and second-price sequential auctions, which implies that in the case of private values winners will be ordered in decreasing order of value. This effect might produce decreasing prices in sequential auctions. However, we will also see that the bidding strategy has the property that the bidder bids more aggressively in further rounds, due to the fact that there are fewer units left for sale. This effect works in the opposite direction and induces an increasing price sequence. In equilibrium these two effects offset each other and prices should have no trend. This result is known as the martingale property of prices. In the case of interdependent values the effect of more aggressive bidding dominates, since bidders can infer signals of previous round-winners and reduce the winner's curse. In that case prices have an increasing trend.

Many empirical papers aiming to confirm the martingale property of prices followed the theoretical analysis of sequential auctions. Surprisingly, the

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<sup>1</sup> In our previous paper, Trifunović and Ristić (2013), we presented some forms of multi-unit auction that are used in the procurement of electricity.

majority of these papers identified the price sequence in sequential auctions to actually be decreasing, which is referred to in the literature as the declining price anomaly. This has motivated a new wave of theoretical research aiming at explaining this phenomenon. The most developed argumentation is based on the assumption that bidders are risk-averse (possess concave utility functions) or that they are averse to price risk. The second stream of research supposes that declining prices might result from the fact that the number of objects that are sold in auctions is uncertain. In this case, if it turns out that the supply is larger than bidders expected, prices will decline. The third explanation of declining prices is based on the assumption that objects are stochastically equivalent, when prices decline if bidders' values are uniformly distributed. The fourth avenue of research focuses on the buyer's option when the winner of the first auction has the possibility of buying the remaining units at the same price he won the first unit. When buyer's option is available, bidders bid more aggressively to obtain this option. Finally, declining prices in English auctions might stem from the behaviour of absent bidders, who bid high since they do not participate in the auction process and have no feedback concerning the intensity of the competition. Moreover, declining price anomaly in sequential English auctions could result from the presence of participation fees when the participation is endogenous.

In the case of interdependent values, the effect of more aggressive bidding dominates the effect of declining values and the price sequence is increasing. However, if an increasing price pattern is empirically observed in the case of independent values or in the case of heterogeneous objects with random assignment, this effect might also be considered as an increasing price anomaly, since it is unexpected from the theoretical point of view.

Finally, in some auctions more complex price patterns are observed when the price sequence is non-monotonic. The possible theoretical explanation for this result might be sought in the decomposition of the price change in two effects: aversion to price risk and information externality effect. When the first effect dominates prices decline, whereas when the second effect dominates prices increase.

The rest of the paper is organised as follows. In the second part we derive the optimal bidding strategy in a sequential first-price auction. The third part deals with the derivation of bidding strategy in a sequential second-price auction with single and multi-unit demand. The property of these strategies is reflected in the fact that price sequence follows a martingale, and this is the subject of the analysis in the fourth part. The fifth part deals with empirical and theoretical aspects of declining price anomaly, while the sixth and seventh parts deal with increasing price anomaly and non-monotonic prices. In the last part we conclude our discussion.

## **2. OPTIMAL BIDDING STRATEGY IN SEQUENTIAL FIRST-PRICE AUCTIONS**

Optimal bidding strategy in sequential first-price auctions can be derived in a recursive manner by starting from the last auction and proceeding backward. We will see that the strategy is strictly increasing. In other words, the amount of bid shading decreases from one auction to the other. We will derive equilibrium strategies in a sequential first-price auction by using a heuristic derivation<sup>2</sup>. Suppose that the bidder has value  $v$  that is distributed according to some probability distribution function  $F(\cdot)$ . In order to grasp some intuition about the nature of bidding strategies, we will start with ordinary first-price auctions. We know that in an ordinary first-price auction the bidder's expected payment equals the expected value of the second highest value<sup>3</sup>:

$$P(v) = \int_0^v y dG(y), \quad (1)$$

where  $G(y) \equiv F(y)^{N-1}$  is the distribution function of the highest order statistics among  $N-1$  values, or the distribution function of the highest value of the bidder's competitors (the distribution function of the second highest value).

The bidder's expected payment in a single first-price auction equals his bid multiplied by the probability of winning  $P(v) = b(v)G(v)$ , and by using this result in (1) it is straightforward to conclude that the optimal bidding strategy is:

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<sup>2</sup> The derivation of equilibrium strategies is a simplified version of the Krishna's proof (2002).

<sup>3</sup> For the derivation of this result, see Trifunović (2010).

$$b(v) = \frac{1}{G(v)} \int_0^v y dG(y). \quad (2)$$

We will use this result to derive equilibrium bidding strategies in sequential first-price auctions, with  $K$  objects, and  $N$  bidders having single-unit demand. When bidders have single-unit demand, the bidder that wins in one auction does not participate in the following auctions. We will analyse the case of multi-unit demand in sequential second-price auctions later on and we will see that the derivation of equilibrium strategies is little bit more involved. We will proceed by using the backward induction to determine the equilibrium strategy in the last  $K$ -th auction. If bidding strategies are strictly increasing, which we will soon see is the case, the allocation will be efficient and bidders with  $K-1$  highest values win the previous auctions. The bidder with value  $v$  will win the  $K$ -th auction when other  $N-K$  still active bidders have lower values. In other words, the bidder with value  $v$  wins the  $K$ -th auction with probability  $G(v) \equiv F(v)^{N-K}$ . By using the analogy with (1), we can conclude that the bidder's expected payment in the  $K$ -th auction will be equal to the expected value of the  $K+1$  highest value conditional on winning or the expected value of the highest of  $N-K$  values conditional on winning. Thus, the bidder's optimal strategy in the  $K$ -th auction is:

$$b_K^I(v) = \frac{1}{F(v)^{N-K}} \int_0^v y d(F(y)^{N-K}), \quad (3)$$

where  $I$  indicates first-price auction. Now, observe how the previous bidding strategy can be written in a different form. The probability that the bidder wins the  $K$ -th auction represents the probability that his value is higher than the  $K$ -th highest of  $N-1$  values,  $Y_K$ , conditional on the event that  $Y_{K-1}$ , and the  $K-1$ -th highest of  $N-1$  values is equal to  $y_{K-1}$ . But since values are independently distributed, this probability is the same as the probability that the highest of  $N-K$  values,  $Y_1^{(N-K)}$ , is less than  $v$ , conditional on  $Y_1^{(N-K)} < y_{K-1}$ . By using this argument, the bidding strategy in the  $K$ -th auction can be written as:

$$b_K^I(v) = E[Y_1^{(N-K)} | Y_1^{(N-K)} < v] = E[Y_K | Y_K < v < Y_{K-1}]. \quad (4)$$

It is important to note that the bidding strategy does not depend on prices from previous auctions.

We will take one step backward and determine the optimal strategy in the  $K-1$ -th auction. The bidder with value  $v$  wins the  $K-1$ -th auction if his value is higher than values of other  $N-K+1$  active bidders, which implies that he wins with probability  $G(v) \equiv F(v)^{N-K+1}$ . If the bidder underbids and loses the  $K-1$ -th auction he has the option of winning the  $K$ -th auction, and his expected payment in the  $K-1$ -th auction will be equal to his expected bid in the  $K$ -th auction. By using this argument, the bidder's optimal strategy in the  $K-1$ -th auction could be written as:

$$b_{K-1}^I(v) = \frac{1}{F(v)^{N-K+1}} \int_0^v b_K^I(y) d(F(y)^{N-K+1}). \quad (5)$$

By applying recursive reasoning, we obtain the bidder's strategy in any  $k$ -th auction:

$$b_k^I(v) = \frac{1}{F(v)^{N-k}} \int_0^v b_{k+1}^I(y) d(F(y)^{N-k}). \quad (6)$$

By using the same argument as in (4), the bidding strategy can be written as:

$$b_k^I(v) = E[b_{k+1}^I(Y_1^{(N-k)} | Y_1^{(N-k)} < v)] = E[b_{k+1}^I(Y_k) | Y_k < v < Y_{k-1}]. \quad (7)$$

When we substitute  $k$  with  $K-1$  in (7) we obtain:

$$b_{K-1}^I(v) = E[b_K^I(Y_{K-1}) | Y_{K-1} < v < Y_{K-2}]. \quad (8)$$

By inserting (4) in the last equation, we obtain:

$$b_{K-1}^I(v) = E[E[Y_K | Y_K < Y_{K-1}] | Y_{K-1} < v < Y_{K-2}] = E[Y_K | Y_{K-1} < v < Y_{K-2}], \quad (9)$$

where the last equality follows from the law of iterated expectations. If we insert this result in (7) and solve recursively, we obtain the equilibrium bidding strategy for auction  $k$ :

$$b_k^I(v) = E[b_{k+1}^I(Y_k) | Y_k < v < Y_{k-1}] = E[E[Y_K | Y_{k+1} < Y_k] | Y_k < v < Y_{k-1}] = E[Y_K | Y_k < v < Y_{k-1}]. \quad (10)$$

To obtain the intuition about these strategies, we will use an example with uniformly distributed values (Krishna, 2003). Assume that each bidder has a private, independently distributed value on the interval  $[0,1]$ . We will first derive the bidder's strategy in the last  $K$ -th auction. In the case of uniform distribution  $F(v) = v$  and from (3) we obtain that:

$$b_K^I(v) = \frac{1}{v^{N-K}} (N-K) \int_0^v y \cdot y^{N-K-1} dy = \frac{(N-K)}{v^{N-K}} \int_0^v y^{N-K} dy = \frac{N-K}{N-K+1} \cdot \frac{y^{N-K+1}}{v^{N-K}} \Big|_0^v = \frac{N-K}{N-K+1} \cdot v. \quad (11)$$

From (5) and (11) we can derive the optimal strategy in the  $K-1$ -th auction:

$$b_{K-1}^I(v) = \frac{1}{v^{N-K+1}} \cdot \frac{N-K}{N-K+1} \cdot \int_0^v y d(y^{N-K+1}) = \frac{1}{v^{N-K+1}} \cdot \frac{N-K}{N-K+1} \cdot (N-K+1) \cdot \int_0^v y^{N-K+1} dy. \quad (12)$$

$$b_{K-1}^I(v) = \frac{1}{v^{N-K+1}} \cdot \frac{N-K}{N-K+2} \cdot v^{N-K+2} = \frac{N-K}{N-K+2} \cdot v. \quad (13)$$

By applying induction we conclude that:

$$b_{K-2}^I(v) = \frac{N-K}{N-K+3} \cdot v, \quad (14)$$

and:

$$b_{K-m}^I(v) = \frac{N-K}{N-K+(m+1)} \cdot v. \quad (15)$$

If we substitute  $K-m \equiv k$ , we obtain from (15) that:

$$b_k^I(v) = \frac{N-K}{N-k+1} \cdot v. \quad (16)$$

It is obvious that the bidding strategy is an increasing function of  $k$ , which means that the bidder bids more aggressively in auctions that are closer to the final auction. For example, assume that  $N = 5$  bidders participate in an auction

for  $K = 3$  objects. From (16) it follows that bidding strategies in these three auctions are:

$$b_1^I(v) = \frac{2}{5} \cdot v, \quad b_2^I(v) = \frac{1}{2} \cdot v \quad \text{and} \quad b_3^I(v) = \frac{2}{3} \cdot v. \quad (17)$$

This example shows us that even in the last auction the bidder bids lower than his value, but he bids more aggressively in subsequent auctions due to the fact that supply becomes more scarce compared to demand. Since the bidding strategy is linearly increasing in value, the winner in the next auction has a lower value than the winner of the previous auction. We will prove later on that these two effects exactly offset each other in equilibrium and the prices have no trend. In other words, the price process is a martingale.

### **3. OPTIMAL BIDDING STRATEGY IN SEQUENTIAL SECOND-PRICE AUCTIONS**

In this part we will derive bidding strategies in sequential second-price auctions with single and multi-unit demand. We will see that the bidder bids more aggressively in sequential second-price auctions than in sequential first-price auctions. Concerning the case with multi-unit demand, we will confine ourselves to the case of two-unit demand where the bidders' valuations exhibit positive or negative synergy.

#### **3.1. Single-unit demand**

We know that in sequential second-price auctions the bidder with the highest value wins the auction and pays the price equal to the second highest bid. The last sequential auction is in fact an ordinary second-price auction and the bidder's optimal strategy is to submit a bid equal to his value. However, in the next-to-last auction the bidder shades his bid, since he has a chance to win the next auction if he loses the current auction. This argument extends to all non-final auctions where bidders shade their bids. The amount of bid shading is equal to the expected profit that the bidder can obtain in the next auction. Bidders bid more aggressively in sequential second-price auctions than in first-price auctions since the winner's bid does not set the price he pays. Formally, Milgrom and Weber (2000) prove that the bidder's strategy in a  $k$ -th second-price auction corresponds to the bidder's strategy in a  $k+1$ -th first-price auction.

As in sequential first-price auctions with private values, prices in sequential second-price auctions also follow a martingale.

As we have argued before, in the  $K$ -th auction the bidder submits a bid equal to his value. In any other  $k$ -th second-price auction the bidding strategy corresponds to the bidding strategy in a  $k+1$ -th first-price auction:

$$b_k^{\text{II}}(v) = b_{k+1}^{\text{I}}(v), \quad (18)$$

where I and II represent first- and second-price auctions. In order to prove this result we will use the fact that both auction mechanisms allocate efficiently, since bidding strategies are strictly increasing. According to the revenue equivalence theorem, auction mechanisms that allocate efficiently and provide the same expected profit to the bidder with the lowest valuation, result in the same expected revenue for the seller. But here we need a more rigorous result. Namely, the expected revenue of the seller in every  $k$ -th first-price auction has to be equal to his expected revenue in a  $k$ -th second-price auction. First, note that in both auction formats bidders have the same information concerning the previous prices, and, since the strategies are symmetric and strictly increasing, they can infer the values of previous winners. Therefore, revenue equivalence implies that the expected revenue for the seller is the same stage by stage.

In order to derive equilibrium bidding strategies in a sequential second-price auction, note that the bidder in the last stage bids as in an ordinary single-unit second-price auction; i.e., he bids his true value for the object:

$$b_k^{\text{II}}(v) = v. \quad (19)$$

In any  $k$ -th auction the bidder with value  $v$  wins with probability  $\Pr[Y_k < v < Y_{k-1}]$  and he pays the price equal to the highest losing bid  $b_k^{\text{II}}(Y_k)$ . Thus, the expected payment of bidder with value  $v$  in the  $k$ -th auction is:

$$P_k^{\text{II}}(v) = \Pr[Y_k < v < Y_{k-1}] \times E[b_k^{\text{II}}(Y_k) | Y_k < v < Y_{k-1}]. \quad (20)$$

In the first-price auction the bidder wins with the same probability, but pays his bid, and his expected payment is:

$$P_k^I(v) = \Pr[Y_k < v < Y_{k-1}] \times b_k^I(v) = \Pr[Y_k < v < Y_{k-1}] \times E[b_{k+1}^I(Y_k) | Y_k < v < Y_{k-1}], \quad (21)$$

where the last equality follows from (7).

From the fact that expected payments in the two auction formats are equal across stages  $P_k^I(v) = P_k^{II}(v)$ , it follows that:

$$\Pr[Y_k < v < Y_{k-1}] \times E[b_k^{II}(Y_k) | Y_k < v < Y_{k-1}] = \Pr[Y_k < v < Y_{k-1}] \times E[b_{k+1}^I(Y_k) | Y_k < v < Y_{k-1}] \quad (22)$$

From the last equation it is straightforward to conclude that the bidder bids more aggressively in the second-price than in the first-price sequential auction. Namely, the bidding strategy in the  $k$ -th second-price auction corresponds to the bidding strategy in the  $k+1$ -th first-price auction:

$$b_k^{II}(v) = b_{k+1}^I(v). \quad (23)$$

By using this result we can return to our example with uniformly distributed values and determine the bidding strategies in a sequential second-price auction:

$$b_k^{II}(v) = v, \quad (24)$$

$$b_k^{II}(v) = b_{k+1}^I(v) = \frac{N-K}{N-(k+1)+1} \cdot v = \frac{N-K}{N-k} \cdot v. \quad (25)$$

If we assume that  $N=5$  and  $K=3$  as before, the bidding strategies in sequential second-price auctions become:

$$b_1^{II}(v) = \frac{1}{2} \cdot v, \quad b_2^{II}(v) = \frac{2}{3} \cdot v \quad \text{i} \quad b_3^{II}(v) = v. \quad (26)$$

An interesting analysis of the optimal order of sale in sequential second-price auctions of private value objects with  $N$  bidders having single-unit demand is given in Bernhardt and Scoones (1994). They suppose that the dispersion of values for object A is greater than the dispersion of values for object B. To simplify the analysis, they suppose that the distribution function for object B is

degenerate and that all bidders have the same value. The optimal strategy calls for the bidder to reduce his bid by the amount of expected profit in the second auction. If the seller first sells object B, the bidders will reduce the bid from the common value. The winner of the first auction will not participate in the second auction. If this bidder has the highest or the second highest value for object A, the price for this object will be lower than in the case when this object is sold first. If the order of sale is reversed and object A is sold first, the bidder who obtains object B in the second auction will have a profit of 0, which implies that there will be no bid shading in the first auction where object A is sold. These results show that the seller maximises his revenue when object A is sold first. Therefore, Bernhardt and Scoones (1994) conclude that in sequential second-price auctions it is optimal to sell the object with the higher dispersion of values first.

### **3.2. Sequential second-price auctions with multi-unit demand**

In contrast to Milgrom and Weber (2000), who assumed that bidders have single-unit demand, Katzman (1999) analyses sequential second-price auctions with multi-unit demand when the bidder who won one auction participates in other auctions. In this case bidders are symmetric in the first auction, but in the next auction the winner of the first auction is asymmetric compared to other bidders. The elegant way to avoid the problem of asymmetry is to use a model with only two sequential auctions. In the second auction the bidder submits a bid equal to his value and the problem of asymmetry does not exist. In this model there are two bidders with two-unit demand. If one bidder obtains two units, the price in the second auction will be higher than in the first auction. On the other hand, if each bidder obtains one unit the price will be constant. Katzman (1999) proves that if the distribution function of value for one bidder dominates the distribution function of the other bidder according to first-order stochastic dominance, the price sequence could be declining. Therefore, the declining price anomaly could be explained by the fact that bidders have multi-unit demand.

We will now derive bidding strategies in the tractable case of two sequential second-price auctions with  $N$  bidders having two-unit demand<sup>4</sup>. We will also

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<sup>4</sup> The derivation of the bidder's optimal strategy in this case is due to Menezes and Monteiro (2004).

assume that there might be positive or negative synergy between the two objects. Specifically, the bidder attaches value  $v$  to the one unit and value  $\alpha v$  to two units, where the parameter  $\alpha$  captures the synergy effect. If  $\alpha > 2$ , the bidder values two units more than twice the value of a single unit. In the case of negative synergy, when  $1 < \alpha < 2$ , the bidder values two units less than twice the value of a single unit. Finally, when  $\alpha = 1$ , the second unit is valueless, and we have the case of single unit demand. For ease of exposition we can normalise the synergy parameter such that  $\theta = \alpha - 1$ , and positive (negative) synergy exists when  $\theta > (<)1$ .

We will derive equilibrium strategies by using backward induction, as before. If the bidder does not win one unit in the first auction, he bids his value in the second auction  $b_2^H(v) = v$ . If he wins one unit in the first auction, he submits a bid equal to the differential value that he assigns to the second unit  $b_2^H(v) = \alpha v - v = \theta v$ .

The derivation of equilibrium strategy in the first auction is more tedious. We will derive the strategy for bidder  $i$  assuming that other bidders  $2, \dots, N$  bid according to the strictly increasing strategy  $b_1^H(v)$  in the first auction. We will denote by  $Y$  the highest order statistics among other bidders' values  $Y = \max\{v_j, j \neq i\}$ , and by  $Y_2$  the second highest value. Suppose that bidder  $i$  has value  $v$  for the first unit, but bids as if his value is  $z$ . The expected profit of this bidder is:

$$\Pi(z) = E[(v - b_1^H(Y) + (\theta v - Y)^+) I_{z > Y}] + E[(v - \max\{\theta Y, Y_2\})^+] I_{z < Y}. \quad (27)$$

The first term on the right-hand side of (27) represents the bidder's expected profit if  $z$  is higher than the second highest value when the bidder wins the first auction and pays the second highest bid  $b_1^H(Y)$ . In that case, he bids  $\theta v$  in the second auction. In the case of positive synergy he will win the second unit as well, whereas in the case of negative synergy he might lose the second auction if  $\theta v < Y$  when the term  $(\theta v - Y)^+$  is equal to 0.  $I_{z > Y}$  represents the indicator variable that has a value of 1 if  $z > Y$  and 0 otherwise. Thus, the first term on the right-hand side of (27) is relevant only when  $z > Y$ . However, if  $z < Y$ , bidder  $i$  loses the first auction and the second term on the right hand-side of

(27) is relevant. In this case bidder with value  $Y$  wins the first auction and bids  $\theta Y$  in the second auction. Other bidders submit bids equal to their value. Therefore, if bidder  $i$  wins the second auction he pays the second highest bid, which is either  $\theta Y$  if the bidder who won the first auction submits the second highest bid, or  $Y_2$  if the bidder with the second highest value among other bidders who compete with bidder  $i$  submits the second highest bid. Note that in the case of positive synergy  $\theta Y \geq Y \geq Y_2$  and  $\max\{\theta Y, Y_2\} = \theta Y$ , whereas only in the case of negative synergy might  $\max\{\theta Y, Y_2\} = Y_2$ .

We can write the expected value of the second term on the right hand side of (27) conditional on  $Y = y$  as:

$$\psi(y) = E[(v - \max\{\theta Y, Y_2\})^+ | Y = y]. \quad (28)$$

We can now write the expected profit of bidder  $i$  as:

$$\Pi(z) = \int_0^z (v - b_1^H(y) + (\theta v - y)^+) f(y) dy + \int_z^1 \psi(y) f(y) dy. \quad (29)$$

By using the Leibnitz rule, we obtain the following first-order condition:

$$\frac{\partial \Pi(z)}{\partial z} = \{v - b_1^H(z) + (\theta v - z)^+ - \psi(z)\} f(z). \quad (30)$$

If the bidder bids according to his true value  $z = v$ , then  $\Pi'(z) = 0$  and:

$$b_1^H(v) = v + (\theta - 1)^+ v - \psi(v). \quad (31)$$

The proof that it is optimal for the bidder to bid according to his true value is given in the appendix.

Therefore, the bidder's optimal strategy in the first auction is:

$$b_1^H(v) = v + (\theta - 1)^+ v - E[(v - \max\{\theta Y, Y_2\})^+ | Y = v]. \quad (32)$$

In the case of positive synergy,  $E[(v - \max\{\theta Y, Y_2\})^+ | Y = v] = 0$ , and  $b_1^{II}(v) = \theta v$ . In the case of negative synergy  $(\theta - 1)^+ v = 0$  and  $b_1^{II}(v) = E[\max\{\theta Y, Y_2\} | Y = v]$ . When the bidder demands only one unit, then  $\theta = 0$ , and the last conditional expectation reduces to  $b_1^{II}(v) = E[Y_2 | Y = v]$ .

Finally, note that in the case of positive synergy the bidder with the highest value wins two units, resulting in efficient allocation, whereas in the case of negative synergy the bidder with the highest value might not win two units.

#### **4. THE MARTINGALE PROPERTY OF PRICES**

In sequential first-price auctions bidders submit sealed bids and the winner pays the price equal to his bid. We have proved that in the case of symmetric bidders with private values the bidding strategy in sequential first-price auctions is strictly increasing in value, which implies that the allocation in this auction is efficient. Due to the fact that in each subsequent auction supply becomes more scarce compared to demand, the bidder bids more aggressively. In other words, the bidder shades his bid even in the last auction but the amount of bid shading decreases from one auction to another. On the other hand, the winner of the  $k+1$ -th auction has a lower value than the winner of the  $k$ -th auction, due to the fact that the bidding strategy is strictly increasing. This effect operates in the opposite direction to the effect of more aggressive bidding and decreases the winning bid in subsequent auctions. Milgrom and Weber (2000) prove that these two effects exactly offset each other and prices in sequential auctions have no trend. In other words, the expected price in the current auction is equal to the realised price in the previous auction. This property of price sequence is called the martingale property of prices. The corollary of this result is that prices cannot have a decreasing trend, since this would offer the opportunity for inter-temporal arbitrage, meaning that it would be profitable to postpone participation to later auctions when the price has declined.

We will now prove formally that equilibrium prices in sequential first-price auctions follow a martingale. We will denote by  $P_k$  and  $P_{k+1}$  random variables that represent prices in auctions  $k$  and  $k+1$ . The realised price in auction  $k$  is

equal to the winning bid in that auction  $p_k = b_k^I(v)$ . From the perspective of auction  $k$ , the price in the next auction is a random variable, and from (7) the expected price in the next auction conditional on the realised price in this auction is:

$$E[P_{k+1} | P_k = p_k] = E[b_{k+1}^I(Y_k) | Y_k < v < Y_{k-1}] = b_k^I(v) = p_k. \quad (33)$$

Milgrom and Weber (2000) also analyse the case of interdependent values and prove that the price sequence is increasing. In this case the bidding strategy is a strictly increasing function of the signal, and after each auction the bidder can invert the bidding strategy of the winner and infer his signal. The revelation of signals reduces the winner's curse and bidders bid more aggressively<sup>5</sup>. In this case the effect of more aggressive bidding dominates the effect of declining values and prices have an increasing trend. In other words, the price process is a submartingale, or formally  $E[P_{k+1} | P_k = p_k] > p_k$ .

Moreover, in the case of interdependent values, the sequential first-price auction yields higher revenue than the discriminatory auction, due to the fact that in sequential auctions the bidder can infer signals of the previous round winners, which reduces the winner's curse and leads to more aggressive bidding. For the same reason the sequential second-price auction yields higher expected revenue than the uniform-price auction.

Previous theoretical results imply that in the case of private values the price sequence is a martingale, and in the case of interdependent values the price sequence is a submartingale. However, most empirical research shows that prices in sequential auctions are declining (supermartingale), which is called declining price anomaly.

## 5. DECLINING PRICE ANOMALY

In this part we will present results of empirical research that have identified declining prices as well as theoretical and experimental explanations for this anomaly. Concerning the theoretical models, we will deal with risk-averse

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<sup>5</sup> Auctions with interdependent values were analysed in more detail in Trifunović (2011).

bidders, uncertain supply, stochastically equivalent objects, buyer's option, and sequential English auctions.

**5.1. Empirical evidence**

As we mentioned before, theoretical models suggest that prices in sequential auctions should follow a martingale in the case of private values or have an increasing trend in the case of interdependent values. However, numerous empirical studies identify a large number of auctions with declining prices. Ashenfelter and Graddy (2003) find that empirical literature relating to declining prices is more abundant than literature related to increasing prices.

The first evidence of declining prices was provided by Ashenfelter (1989), who examines wine auctions and determines the probability that the price remains the same, increases, or decreases from one auction to the other. He finds that the highest likelihood is that the price remains the same in the next auction, but it is twice more likely that the price decreases than increases<sup>6</sup>. The results of this empirical research are shown in the following table.

**Table 1:** *The prices of wine in sequential auctions* (Ashenfelter, 1989)

|                        | <i>Christie's<br/>London</i> | <i>Sotheby's<br/>London</i> | <i>Christie's<br/>Chicago</i> | <i>Butterfield's<br/>San Francisco</i> |
|------------------------|------------------------------|-----------------------------|-------------------------------|--|
| Price increase         | 11.43%                       | 8.69%                       | 18.04%                        | 20%                                    |
| Price decline          | 26.5%                        | 26.12%                      | 36.67%                        | 41%                                    |
| Constant price         | 63.2%                        | 65.19%                      | 45.29%                        | 39%                                    |
| Second/first price     | 0.9943                       | 0.9875                      | 0.9884                        | 0.9663                                 |
| Standard error         | 0.00128                      | 0.00188                     | 0.00335                       | 0.0103                                 |
| Number of observations | 2370                         | 1646                        | 499                           | 100                                    |

McAfee and Vincent (1993) also perform empirical analysis of wine auctions at Christie's and determine that the average price in the second auction is 1.4%

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<sup>6</sup> Begs and Grady (1997) identify a similar price pattern in art auctions.

lower than in the first auction. Ashenfelter and Genesove (1994) study sequential real-estate auctions. Some of the apartments that were not sold through the auction mechanism were sold after the auction in face-to-face bargaining. Also, some of the apartments obtained in the auction were later resold in the face-to-face bargaining. The most important conclusion of their empirical analysis is that prices decline during the auction, confirming the price anomaly. The prices decline on average by 0.27% from one auction to another and by 10% from the first to the last auction. However, they intend to discern the effect of declining prices due to the auction format from the effect of declining prices due to the lower quality of units sold later in the auction process. Some of the units obtained in the auction were later resold, generally at a lower price. Ashenfelter and Genesove (1994) find a higher discount for units sold earlier than for units sold later in the auction. This indicates that bidders paid a premium for the units sold earlier in the auction and the declining price anomaly was the result of the auction mechanism. In fact, they determine that, at most, 25% of the price decline was related to quality difference.

Ginsburgh and Ours (2006) study auctions of Chinese porcelain found in ships that sank 150 to 300 years ago. The porcelain from three ships was sold in three separate sequential auctions at Christie's. They analyse only the sale of homogenous objects grouped in different lots. Their research has two important conclusions. First, the seller obtains higher revenue when he sells lots with an identical number of items than when he sells lots with a different number of items. This is confirmed by the actual seller's behaviour, since the seller increased the proportion of lots with an identical number of items from 17% in the first sequential auction to 48% in the second sequential auction and to 68% in the third sequential auction. This result stems from the fact that the constant lot size generates more intense competition. Second, prices drift downward. The price decline is faster for larger lots and the price decline between two successive lots becomes smaller with the number of lots remaining. In contrast to this result, Van den Berg, van Ours, and Pradhan (2001) determine that in sequential Dutch rose auctions the fewer the number of remaining units the larger the price decline.

Thiel and Petry (1995) found evidence of declining price anomaly in sequential second-price auctions in the sale of rare stamps between 1923-1937. All these

empirical results identifying the declining price anomaly have motivated theoretical research aimed at explaining this phenomenon.

## **5.2. Theoretical explanations**

In this part we will present several theoretical explanations of the declining price anomaly. The most developed argumentation is based on bidder's risk aversion. Alternative explanations use the assumptions that the supply is uncertain, that objects are stochastically equivalent, or that the presence of buyer's option induces declining prices. Finally, we will examine theoretical explanations of declining prices in sequential English auctions.

### **5.2.1. Risk aversion**

The first idea that declining prices in sequential auctions might result from the bidder's risk aversion was presented in Buccola's study of livestock auctions (1982). Buccola (1982) claims that the most risk-averse bidders are willing to pay more to win first auctions, and that winners are ordered according to their risk aversion, which induces a downward trend in prices in sequential auctions<sup>7</sup>. By developing Ashenfelter's (1989) idea, McAfee and Vincent (1993) provide a theoretical model that explains declining prices by using the assumption that bidders are risk-averse. Ashenfelter (1989) claims that the expected price in the first auction is equal to the sum of the expected price in the second auction and the risk premium for the price uncertainty in the second auction. Thus, the expected price in the second auction is lower than the expected price in the first auction and the difference in expected prices is equal to the risk premium. In order to obtain equilibrium bidding strategies, McAfee and Vincent (1993) assume that bidders have non-decreasing absolute risk aversion. If bidders have decreasing absolute risk aversion, which is an empirically justified assumption, the equilibrium does not exist. This is the most important drawback of this model since it is based on the empirically implausible behaviour of bidders' absolute risk aversion.

In contrast to the previous model where bidders have a concave utility function, Mezzeti (2011) assumes that the declining-price anomaly stems from the

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<sup>7</sup> However, Buccola's (1982) study was research primarily from the agricultural point of view and the first economic research of declining price anomaly was presented in Ashenfelter (1989).

aversion to price risk. Maskin and Riley (1984) first introduced this type of risk aversion in the context of optimal auction design. The advantage of this approach is its tractability. The classical models of risk-averse bidders with concave utility function are intractable, and, as we mentioned before, in the model of McAfee and Vincent (1993) the equilibrium does not exist when bidders possess decreasing absolute risk aversion.

Mezzeti (2011) uses the general form of the payoff function that applies to both private and interdependent values. In this setup, the payoff function has the form:

$$\Pi(v_i, v_{-i}) = V(v_i, v_{-i}) - l(p), \quad (34)$$

where  $v_i$  represents the signal of bidder  $i$  in the case of interdependent values, and his value in the case of private values, and  $v_{-i}$  is the vector of signals (values) of competing bidders. In the case of private values, the valuation of the object of bidder  $i$  does not depend on  $v_{-i}$ .  $l(p)$  is a convex loss function that captures aversion to price risk. The inverse of the loss function,  $\phi = l^{-1}$ , is strictly increasing and concave. The most convincing hypothesis justifying the convex cost function is that bidders are financially constrained and must borrow, with the cost of borrowing nonlinearly increasing with the amount demanded.

Using the similar derivation as in the case of risk-neutral bidders, the bidding function in sequential first-price auctions with private values is:

$$b_k^I(v, y_{k-1}, \dots, y_1) = b_k^I(v) = \phi(E[Y_K | Y_k < v < Y_{k-1}]), \quad (35)$$

where  $Y_k$  is, as before, the  $k$ -th highest of  $N-1$  values, and  $y_{k-1}, \dots, y_1$  are realisations of the values of bidders who won previous auctions. As in the previous case, this bidding strategy is independent of price history. In the case of

sequential second-price auctions with private values, the bidding strategy could be written in the following form<sup>8</sup>:

$$b_k^{\prime\prime}(v, y_{k-1}, \dots, y_1) = b_k^{\prime\prime}(v) = \phi(E[Y_K | Y_k = v]). \quad (36)$$

When bidders are averse to price risk, the loss function is actually a martingale, and the price sequence in a sequential first-price auction is a supermartingale. This result can be proved in the following fashion. If the bidder with value  $v$  wins the  $k$ -th auction, then  $Y_K \leq v \leq y_{k-1} < \dots < y_1$ . In the first-price auction the price is determined with the winning bid,  $P_k^I = b_k^I(v) = \phi(E[Y_K | Y_k < v < Y_{k-1}])$ . Thus, the conditional expectation of the next period price could be written as:

$$\begin{aligned} E[P_{k+1} | P_k = p_k] &= E[P_{k+1} | b_k^I(v)] = E[b_{k+1}^I(Y_k) | Y_k < v < Y_{k-1}] = \\ &= E[\phi(l(b_{k+1}^I(Y_k))) | Y_k < v < Y_{k-1}] < \phi(E[l(b_{k+1}^I(Y_k))] | Y_k < v < Y_{k-1}) = \\ &= \phi(E[Y_K | Y_k < v < Y_{k-1}]) = b_k^I(v) = p_k. \end{aligned} \quad (37)$$

The inequality results from the Jensen's inequality and the fact that  $\phi$  is a concave function. This result shows us that declining prices result from aversion to price risk. The intuition for this result is that the bidder must be equally indifferent to a certain loss in the current period and an expected loss in the future period. The bidder considers the price in the current period as certain and equal to his bid, whereas the price in the second period is a random variable. When the bidder is equally indifferent to the certain current price and the uncertain price in the next period, the expected price in the next period must be lower than the current price. The difference between the two prices measures the price risk.

It can be proved by using a similar chain of reasoning that the price process in the second-price auction is also a supermartingale:

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<sup>8</sup> We have derived this result in the section dealing with sequential second-price auctions with multi-unit demand for  $K=2$ .

$$\begin{aligned}
 E[P_{k+1} | P_k = p_k] &= E[P_{k+1} | b_k^H(v)] = E[b_{k+1}^H(Y_k) | Y_k = v] = \\
 &= E[\phi(l(b_{k+1}^H(Y_k))) | Y_k = v] < \phi(E[l(b_{k+1}^H(Y_k)) | Y_k = v]) = \\
 &= \phi(E[Y_K | Y_k = v]) = b_k^H(v) = p_k.
 \end{aligned} \tag{38}$$

Mezzeti (2011) then analyses the case of interdependent values with independently distributed signals. In contrast to Milgrom and Weber (2000), who derive the result of increasing prices in the case of affiliated signals, Mezzeti (2011) proves that the price sequence in both first- and second-price sequential auctions is a submartingale (increasing sequence), even when signals are independent. This effect is called the information externality effect.

### 5.2.2. Uncertain supply

The other explanation of the decreasing price anomaly is given in Jeitschko (1999), who considers sequential second-price auctions with uncertain supply when bidders have single-unit demand. The most important result of this paper is that if the number of units for sale is larger than the number that bidders expect, prices will be declining, and if the number of units is smaller than the number of units that bidders expect, prices will increase. Therefore, this model also explains the increasing price sequence. More precisely, in Jeitschko's (1999) model, after the first auction bidders do not know whether there will be one or two more units for sale. This uncertainty induces more aggressive bidding in the first auction. If it turns out that two more units are available, bidders will bid less aggressively in remaining auctions due to the fact that the supply is larger than expected, and if it happens that only one more unit is available, bidders will bid more aggressively in the second auction than in the first auction. This model provides the explanation of declining prices when the supply is uncertain. However, declining prices are observed in auctions with fixed supply and this model does not explain that phenomenon.

### 5.2.3. Stochastically equivalent objects

One more explanation for declining prices is presented in Engelbrecht-Wiggans (1994) who considers the sale of stochastically equivalent objects in sequential first-price auctions. In the standard case of the sale of identical objects, it is logical to assume that the value for the first unit is higher than the value for the

second unit. In the case of stochastically equivalent objects, the value for each unit is a random variable distributed according to some joint probability distribution function. For example, if there are  $N$  bidders with two-object demand, the values that one bidder assigns to two objects  $v_1$  and  $v_2$  are independent random variables distributed according to some joint probability distribution function. One example of an auction of stochastically equivalent objects might be the sale of used furniture. The same pieces of furniture are identical objects when they are sold as new. However, when the same pieces of used furniture are sold, they are preserved to a different extent. The seller chooses randomly the order in which objects are sold, which means that values that bidders assign to these objects are random variables and the values for objects are not arranged in decreasing order.

In the model of Engelbrecht-Wiggans (1994), bidders with single-object demand obtain information before each auction about the object that is to be sold. As we have explained before, there are two opposing effects that influence the bidding strategy. As supply becomes scarcer compared to demand, bidders bid more aggressively, which induces an increasing price sequence. On the other hand, bidders have single-object demand, which imposes the effect of declining values and consequently declining price sequence. Engelbrecht-Wiggans (1994) proves that the first effect dominates when bidders have an exponential distribution function, and prices will have an increasing trend. In the case of uniform distribution the second effect dominates and the price sequence will be declining.

#### **5.2.4. Buyer's option**

In some sequential auctions the winner of the first auction has the opportunity to purchase additional units at a price equal to the price for the first unit. The first model studying the impact of buyer's option on declining price pattern is by Black and de Meza (1992). In their model the price in the first auction is higher than in the second auction, since bidders compete more intensively for the buyer's option to obtain the opportunity to buy subsequent items at the same price.

Gale and Hausch (1994) consider auctions of two stochastically equivalent objects with interdependent values. Gale and Hausch (1994) compare sequential

second-price auctions with and without buyer's option to chose the object. When this option is available, the winner of the first auction has the possibility of choosing either the first or the second object. Gale and Hausch (1994) prove that prices have a decreasing trend in auctions with buyer's option, since bidders bid more aggressively in the first auction in order to obtain the possibility of choosing between the first and second object.

The experimental study of sequential auctions of Février, Linnemer and Visser (2007) examines sequential first-price, second-price, Dutch, and English auctions. They analyse the impact of buyer's option on the declining price anomaly. In the experimental study of Février, Linnemer and Visser (2007) there are two units for sale, and the winner of the first auction can exercise the option to buy the second unit at the same price. If he does not exercise the option, the second unit is sold in the remaining auction. The bidder assigns value  $v$  to the first unit and  $\theta v$  to the second unit, where the parameter  $\theta$  takes three values  $\theta \in \{1/2, 1, 2\}$ . When  $\theta = 1/2$  there is negative synergy, with  $\theta = 1$  there is no synergy, and with  $\theta = 2$  there is positive synergy.

Février, Linnemer and Visser (2007) conducted 12 experimental sessions without and 12 with buyer's option. Their conclusion is that the bidding strategy is close to the equilibrium strategy for the second unit, but differs substantially from the equilibrium strategy for the first unit. Moreover, they find that in the case of positive synergy and without the buyer's option the loser of the first round bids higher than his value in the second round. This phenomenon could be explained by fairness considerations, since losing bidders want to punish their rivals. Concerning the buyer's option, the theory predicts that bidders should use the option in the case of no synergy and positive synergy when the value of the second unit is higher than the price of the first unit. The experimental research confirms this theoretical result and declining price anomaly exists when the buyer's option is available.

### **5.2.5. Sequential English auctions**

Mezzeti (2011) studies sequential English auctions with private values. He claims that the most widely used version of English auction in auction theory, the so-called Japanese version, is not strategically equivalent to real-life sequential English auctions. In the Japanese version the price is constantly

increased on the screen and bidders press the button when they are active and drop out from the auction when the price reaches their value. Once the bidder releases the button he drops out from the auction and he cannot re-enter. Thus, in the Japanese version the values of losing bidders are released in the first sequential auction and there is no price risk in further rounds. Therefore aversion to price risk cannot lead to decreasing prices.

However, in real-life sequential English auctions all bidders are not obliged to participate in every round. It might happen that some bidders remain silent in the first round and enter the auction in some of the following rounds. This creates uncertainty concerning the valuation of these bidders and their competitors now face price risk in further rounds. This implies that in real-life English auctions the price sequence could be declining due to the price risk.

Ginsburgh (1998) provides a similar explanation for declining prices in sequential English wine auctions. In these auctions bidders are often absent and have informed the auctioneer about their bids before the beginning of the auction. The auctioneer bids on behalf of these bidders. Ginsburgh (1998) determines that in 60% of cases all the lots were won by absent bidders, in 30.8% of cases there was competition between present and absent bidders, and in 9.2% of cases only present bidders won all lots. Moreover, in 81.5% of cases the first lot is won by an absent bidder. In the analysed sample declining prices are observed 179 times. In 99 of these 179 cases only absent bidders won, in 79 cases there was competition between present and absent bidders, and only in one case did present bidders win all lots. Thus, the declining price anomaly in wine auctions might be explained by the behaviour of absent bidders who bid high, since they have no awareness of the competition in the auction. However, declining prices are observed in other sequential auctions that operate differently from wine auctions.

Von der Fehr (1994) analyses predatory bidding in sequential English auctions with private values. In his model  $K=2$  and participation in the second stage is endogenous since bidders incur participation costs. As a benchmark case he studies two sequential English auctions without participation costs. In this case in the second auction the bidder's dominant strategy is to stay active until the price reaches his value, and the object is sold at the third highest value (the

second highest value in the second stage). In the first auction no bidder is willing to bid higher than the price in the second auction and the prices are the same. In the other extreme case, where all bidders bear participation costs, the bidder with the third highest value will not participate in the second auction, since bidding strategies in the first auction are fully revealing and he knows that he has no chance of winning the second auction. The loser with the highest value in the first auction will participate in the second auction and he will obtain the object for free. The price in the first auction is equal to the third highest value and the price declines sharply in the second auction. In the general case when some proportion of bidders incurs participation costs and others can compete freely, the declining prices are still present, although this effect is not so severe. In this case the bidder with the third highest value does not participate in the second auction with positive probability (stochastic participation) and the two bidders with the highest values have an incentive for predatory bidding. In other words, bidders with the two highest values stay active above the third highest value in the first auction to deter the bidder with the third highest value from participating in the second auction.

## **6. INCREASING PRICE ANOMALY**

Gandal (1997) provided the first evidence of increasing prices in his study of procurement auctions for Israeli cable TV licenses. Sequential procurement auctions were organised for procurement licenses in 30 regions of the country. The main finding of the paper is that there are substantial interdependencies among the licenses and that subscriber fees decreased during the auction. The phenomenon of declining subscriber fees in procurement auctions could be transposed to increasing price sequences in ordinary auctions.

Raviv (2006) found more evidence of increasing prices in a sequential second-price auction of used cars in New Jersey. In this auction the cars are randomly sold and there is no significant relationship between the estimated value of the car and the order in which it is sold. In other words, this is a sequential auction of heterogeneous objects with random assignment. The empirical results show that the price pattern is increasing. However, prices increase only in the first half of the auction, and then remain constant. Raviv (2006) uses the number of bids as a proxy for the level of competition and concludes that the level of competition increases during the first 12 rounds and after that the bidding

behaviour does not change. Raviv (2006) does not offer a definite conclusion concerning the bidders' valuations. If bidders have private values then the phenomenon of increasing prices could be called increasing price anomaly and is related to the fact that bidders are not warmed up at the beginning of the auction. On the other hand, if values are interdependent the empirical results coincide with the theory. The fact that the price pattern is first increasing and then constant might be interpreted in the following fashion. When the increasing price pattern stops, all the benefits from information revelation have been exhausted and bidders possess all the necessary information regarding the value of the object. However, this effect might stem from the fact that informed bidders shade their bids in the first half of the auction to conceal their private information.

Regardless of the nature of bidders' valuations, used cars that are sold in this auction represent heterogeneous objects. But the increasing price pattern for heterogeneous objects with random assignment is unexpected from the theoretical point of view, and the empirically observed phenomenon of increasing prices could also be considered an anomaly. Raviv (2006) provides an example that implies that the order of sale does not influence the seller's revenue in sequential second-price auctions and that the expected price should remain the same in both rounds when the order of sale of heterogeneous objects is random.

Suppose that 3 bidders compete for 2 heterogeneous objects of high and low quality. The bidder assigns to the object of high quality  $H$  value  $v_i$  and to the object of low quality  $L$  a value of  $t v_i$ , where  $0 < t < 1$ , and bidders' valuations are independent. The random variable  $v_i$  can take a value of 1 with probability  $p$  and a value of 0 with probability  $1-p$ . Thus, the bidder with high valuation assigns a value of 1 to object  $H$  and value  $t$  to object  $L$ , and the bidder with low valuation assigns a value of 0 to both objects. The quality of the good is common knowledge, while the bidder's valuation is his private information.

We will first determine the seller's expected revenue if the high quality good is auctioned first. We will denote by  $F(b)$  the distribution function of bids for the bidder with value 1 for object  $H$ . Note that a bidder with a value of 0 would not participate. The expected profit of bidder with value 1 in this case is:

$$\Pi(b) = (1-p)^2 1 + 2p(1-p) \left[ \int_0^b (1-x) dF(x) + (1-F(b))t \right] + p^2 \left[ \int_0^b (1-x) dF^2(x) \right]. \quad (39)$$

The first term is the expected profit when the other two bidders have a value of 0 and do not participate and the bidder with a value of 1 obtains the object  $H$  for free. The probability of this event is  $(1-p)^2$ . The second term represents the expected profit of the bidder who bids  $b$  when that bidder competes with another bidder with a value of 1 while the other bidder has a value of 0. This event occurs with probability  $2p(1-p)$ . If the bidder wins the first auction, he obtains a net gain equal to the difference between his value and his opponent's bid,  $1-x$ . If he loses the first auction with probability  $1-F(b)$ , he can win the second auction and obtain the object  $L$  at the second auction for free, receiving a gain of  $t$ . The third term represents the expected profit in the case where all 3 bidders have high value and the probability of this event is  $p^2$ . The bidder wins the first auction with probability  $F^2(x)$  and he obtains a net gain of  $1-x$ . If he loses the first auction he will obtain a profit of 0 in the second auction, since his competitor has the same value and submits the same bid.

By differentiating (39) and by using the Leibnitz rule we obtain the first-order condition:

$$2p(1-p)(1-b)f(b) - 2p(1-p)f(b)t + p^2(1-b)2F(b)f(b) = 0. \quad (40)$$

From the first-order condition we can obtain the probability distribution and density function for the bids:

$$F(b) = \frac{(t-(1-b))(1-p)}{p(1-b)}, \quad f(b) = \frac{(1-p)t}{p(1-b)^2}. \quad (41)$$

Since the distribution function must be non-negative, it follows that  $b \geq 1-t$ . Also, the distribution function is bounded between 0 and 1, which implies that the numerator must be lower than the denominator:

$$(t-(1-b))(1-p) \leq p(1-b). \quad (42)$$

This condition reduces to

$$b \leq 1 - (1 - p)t. \tag{43}$$

Thus, the bid of the bidder with value 1 is bounded from below and above  $1 - t \leq b \leq 1 - (1 - p)t$ . In order to determine the expected revenue of the seller, we need to determine the probability density function of the second highest of  $N$  bids. The second-highest bid is lower than  $x$  if all values are lower than  $x$  and the probability of this event is  $F(x)^N$ . The second highest value can also be less than  $x$  in the event that the highest value is higher than  $x$  and the second highest value is lower than  $x$ . The probability of this event is  $N F(x)^{N-1}(1 - F(x))$ . Thus, the distribution function of the second highest bid is:

$$F_2(x) = F(x)^N + N F(x)^{N-1}(1 - F(x)). \tag{44}$$

Differentiating the distribution function, we obtain the probability density function:

$$f_2(x) = N(N - 1)(1 - F(x))F(x)^{N-2}f(x). \tag{45}$$

Using these results we can calculate the expected selling price in the first auction:

$$3p^2(1 - p) \int_{1-t}^{1-(1-p)t} 2x(1 - F(x))dF(x) + p^3 \int_{1-t}^{1-(1-p)t} 6xF(x)(1 - F(x))dF(x). \tag{46}$$

The first term represents the expected selling price when two bidders have high value and one bidder has low value. This event occurs with probability  $3p^2(1 - p)$ . The integral represents the expected value of the second-order statistics for  $N=2$ , where the probability density function is obtained from (45). The second term is the expected selling price when 3 bidders have high value and this happens with probability  $p^3$ . The integral represents the expected value of the second-order statistics for  $N=3$ . Integrating the last expression, we obtain the expected selling price  $p^2(3pt - 3t - 2p + 3)$ .

In the second auction the price is positive and equal to  $t$  only when 3 bidders have high value. Thus, the expected selling price in the second auction is  $p^3t$ .

We will now consider the case when the order of sale is reversed and the object of low quality is sold first. If the bidder with value 1 wins the first auction and pays  $t$ , he can participate in the second auction and bid up to  $1-t$  for object  $H$ . If the bidder with value 1 loses the first auction, he will bid up to 1. We will denote by  $G(b)$  the distribution function of bids. The expected profit of the bidder with value 1 is:

$$\Pi(b) = (1-p)^2 1 + 2p(1-p) \left[ \int_0^b (t-x) dG(x) + (1-G(b))t \right] + p^2 \left[ \int_0^b (t-x) dG^2(x) \right]. \quad (47)$$

The first term is the expected profit when the bidder competes against two bidders with a value of 0 when this bidder pays 0 and obtains object  $L$  that is worth  $t$  to him. He will also participate in the second auction and bid up to  $1-t$ , yielding him a net gain of 1. The second term is the bidder's expected profit when he competes with one bidder with high value and another bidder with low value. If the bidder wins the first auction he pays the second highest bid  $x$  and obtains a profit  $t-x$ . The bidder can lose the first auction with probability  $1-G(b)$ . In that case he will compete with the winner of the first auction in the second stage, who is willing to bid up to  $1-t$ . Since the bidder who lost the first auction is willing to pay 1, he will have a net gain of  $t$ . The third term is the bidder's expected profit when he competes with two other bidders with high value. In this case the bidder's gain from the second auction is 0, regardless of whether he wins or loses that auction. The bidder can obtain a positive gain of  $t-x$  only if he wins the first auction. From the first order condition and from the Leibnitz rule, we obtain the probability distribution and probability density function of bids:

$$G(b) = \frac{b(1-p)}{p(t-b)}, \quad g(b) = \frac{(1-p)t}{p(t-b)^2}. \quad (48)$$

To satisfy the non-negativity of the probability distribution function and the property that it is bounded between 0 and 1, the optimal bid must satisfy  $0 \leq b \leq pt$ .

By using the analogy with (46) the expected selling price in the first auction is:

$$3p^2(1-p)\int_0^{pt} 2x(1-G(x))dG(x) + p^3\int_0^{pt} 6xG(x)(1-G(x))dG(x) = p^3t. \quad (49)$$

The selling price in the second auction is  $1-t$  if there are two bidders with high value and one bidder with a value of 0. This event is realised with probability  $3p^2(1-p)$ . When all 3 bidders have high value, the selling price in the second auction is 1 and the probability of this event is  $p^3$ . Thus, the expected selling price in the second auction is  $3p^2(1-p)(1-t) + p^3 = p^2(3pt - 3t - 2p + 3)$ .

This example demonstrates that the expected price of heterogeneous objects is the same regardless of the order in which they are sold. Moreover, when the order of sale is random and when both objects have an equal chance of being sold in the first auction, the expected selling price is equal in both rounds. This result is in contrast with the result of Beggs and Graddy (1997), who find that the seller maximises expected revenue when he sells heterogeneous objects in decreasing order of quality. Therefore, the empirically observed phenomenon of increasing prices in the auction of used cars in New Jersey that represents auctions of heterogeneous objects with random assignment could be considered an anomaly.

Deltas and Kosmpolou (2004) find evidence of increasing prices in sequential auctions of rare books organised by the University of Illinois. In these auctions the books are sold by alphabetical order of the author's name and no price trend is expected due to differences in the quality of objects. Therefore, if an increasing price trend is observed, it could also be considered an anomaly. The library personnel prepared the description of books in the catalogue. The number of lines describing each book differed. In this auction one group of bidders submitted mail-in bids with a predetermined bid for each lot (mail bidders) while other bidders played an active role in the competition (floor bidders). The increasing price trend in this auction is a consequence of non-strategic bidding by some of the bidders. In fact, bidders submitted more bids for lots with a more detailed description. A lot that was described with 9 lines in the catalogue had a price decline of 2.3% from the first to the last sale. A lot with 14 lines of description experienced a price increase of 6.8%, and a lot described with 21 lines had a price increase of 23.1%. This effect is called the catalogue

effect. Moreover, the catalogue effect is less present in the group of floor bidders than in the group of mail-in bidders.

The second effect that governs the increasing price trend is the order-of-sale effect. This effect can be disentangled from the catalogue effect by analysing lots with a median number-of-lines description. The price of these lots increased 14.5% in auctions with only floor bidders and 6.8% in auctions with mail-in bidders. This result shows clearly that the increasing price trend is an anomaly.

One additional conclusion of this paper is that an increasing price trend is accompanied by a lower probability of sale as the auction progresses. This phenomenon is counterintuitive at first glance, since the increasing price pattern suggests an increasing interest of bidders in participating, while the lower probability of sale indicates a lower interest in participating. Deltas and Kosmpolou (2004) construct a theoretical model aimed at reconciling these two effects and demonstrate that this combination of increasing prices and lower probability of sale can be obtained in equilibrium.

## **7. NON-MONOTONIC PRICE PATTERN**

In the previous discussion we have presented theoretical results that imply that prices in sequential auctions should remain constant in the case of private values or in the case of heterogeneous objects with random assignment. In the case of interdependent values prices should have an increasing trend. We have also presented empirical results that contradict these findings and are referred to in the literature as declining and increasing price anomaly. However, more complicated price patterns are observed in practice. For example, Jones, Menezes and Vella (2004) determine that prices in wool auctions in Australia can go in either direction. The price pattern can be increasing or decreasing during a sequential auction. They claim that a theoretical model has to be able to explain this phenomenon. Mezzeti (2011) present a possible theoretical explanation of this price pattern. Furthermore, Jones, Menezes and Vella (2004) find that the price anomaly is not related to the characteristics of wool and that the existence of an anomaly is more likely in auctions that last longer. However, the sign of the anomaly is independent of this factor.

This unusual price pattern where prices can increase or decrease could be explained in the following way. Mezzeti (2011) proves that the expected price in the next auction conditional on current price can be decomposed into three terms:

$$E[P_{k+1} | P_k = p_k] = P_k + A(v, y_{k-1}, \dots, y_1) + I(v, y_{k-1}, \dots, y_1), \quad (50)$$

where  $P_k$  is the price in the previous auction,  $A(v, y_{k-1}, \dots, y_1)$  is the effect of aversion to price risk, and  $I(v, y_{k-1}, \dots, y_1)$  is the effect of information externalities. This general formulation permits the price sequence to be decreasing, increasing, or constant and justifies more complicated price patterns. We have explained previously how these two effects operate and we know that the effect of aversion to price risk induces an increasing price trend while the effect of information externalities operates in the opposite direction. When the first effect dominates the price sequence is decreasing and when the second effect dominates it is increasing.

## **8. CONCLUSION**

We have presented different forms of sequential auction where bidders have single and multi-unit demands. The bidding strategies in sequential first-price and second-price auctions are strictly increasing and have the property that the bidder bids more aggressively from one auction to the other. On the other hand, the effect of declining values operates in the opposite direction, and in the case of private values prices should have no trend, while in the case of interdependent values prices should have an upward trend. Nevertheless, empirical research shows that prices decline in private value auctions, increase in auctions of heterogeneous objects with random assignment, or have a non-monotonic price pattern. These findings have motivated both theoretical and experimental research that intends to explain these paradoxes. Despite these considerable efforts, there is still no definite answer to these empirical anomalies. It should be added that more experimental research is needed to understand the actual bidder's behaviour and how it departs from Bayesian-Nash strategies. The results of this experimental research should be taken with caution since the bidding strategies are derived under the assumption of symmetry, and in real-life auctions bidders are asymmetric. However, the

derivation of equilibrium strategies in sequential auctions with asymmetric bidders is a tedious task and possibly intractable.

Despite all these controversies, more empirical, theoretical, and experimental research is needed to understand sequential auctions. This further research is important not only from an academic but also from a practical point of view, since sequential auctions are used in the sale of various objects in practice.

## APPENDIX

### Sincere bidding in a sequential second-price auction with multi-unit demand

When we substitute (31) in (30), we obtain:

$$\frac{\Pi'(z)}{f(z)} = v - z - (\theta - 1)^+ z + (\theta v - z)^+ . \quad (A1)$$

In the case of negative synergy the term  $(\theta - 1)^+$  is equal to 0, and (A1) reduces to:

$$\frac{\Pi'(z)}{f(z)} = v - z + (\theta v - z)^+ . \quad (A2)$$

If positive synergy exists, then  $(\theta - 1)^+ > 0$ , and:

$$\frac{\Pi'(z)}{f(z)} = v - \theta z + (\theta v - z)^+ . \quad (A3)$$

We will now show that it is optimal for the bidder to bid according to his true value. First, consider the case of negative synergy  $\theta < 1$ . Note that from (A2) for  $v > z$ ,  $(\theta v - z)^+ > 0$  if  $\theta v > z$  and  $(\theta v - z)^+ = 0$  when  $\theta v < z$ . Therefore,  $\Pi'(z) > 0$ . In the opposite case, when  $v < z$ ,  $(\theta v - z)^+ = 0$  and  $\Pi'(z) < 0$ . This implies that bidder  $i$  maximizes his expected profit when  $v = z$ .

In the case of positive synergy,  $\theta > 1$ ,  $v > z$  implies that  $\theta v > v > z$  and from (A3), by using the fact that  $(\theta v - z)^+ > 0$ , we obtain that:

$$\frac{\Pi'(z)}{f(z)} = v - \theta z + \theta v - z = \alpha(v - z) > 0. \quad (\text{A4})$$

In the opposite case when  $z > v$ , we have two subcases. If  $(\theta v - z)^+ = 0$ , then from (A3) it is obvious that  $\Pi'(z) < 0$ . On the other hand, if  $(\theta v - z)^+ > 0$ , then:

$$\frac{\Pi'(z)}{f(z)} = v - \theta z + \theta v - z = \alpha(v - z) < 0. \quad (\text{A5})$$

Therefore, in the case of positive synergy it is also optimal for the bidder to bid according to his value.

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