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*Dejan Trifunović**

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OPTIMAL AUCTION MECHANISMS WITH PRIVATE VALUES

ABSTRACT: *This paper reviews equilibrium behaviour in different auction mechanisms. We will deal with two types of open auctions, English and Dutch, and two types of sealed-bid auctions, first-price and second-price, when there is a single object for sale and bidders have private values. We show that under certain conditions all four auctions yield the same expected revenue to the seller, but once these assumptions are relaxed revenue equivalence does not hold. We will also study auctions by using standard tools from demand theory. Finally, we will analyse collusive behaviour of bidders. The two goals that an auction mechanism has to achieve are efficient allocation and maximization of the seller's expected revenue.*

KEY WORDS: *English auction; Dutch auction; First-price sealed-bid auction; Second-price sealed-bid auction.*

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* Faculty of Economics, Belgrade, dejan@ekof.bg.ac.rs .This paper was written with the support of the Ministry of Science of the Republic of Serbia, grant number 149041.

1. INTRODUCTION

We are familiar with auctions that are used to sell antiquities or valuable objects. They are also used to sell rights to use natural resources, such as spectrum rights used in telecommunications. In the former communist countries, entire enterprises previously owned by the state are sold in auctions. Government contracts, as well as governments' short-term debt are also sold by means of auction. The Greek historian Herodotus gives the first known description of auctions when he describes the sale of women in ancient Babylon. The design of auctions might seem a trivial task, but it is a very serious and interesting field of economic analysis.

Viliam Vickrey (1961) was the first person to study auctions from the theoretical point of view, and he was awarded the Nobel Prize in Economics in 1996 for his research in this area. Since his pioneering work, the field has grown substantially and in 2007 another Nobel Prize was awarded to Leonid Hurwitz, Roger Myerson and Eric Maskin for their study of *mechanism design*, a field closely related to auctions. Roughly speaking, a mechanism is a set of rules that govern interactions between economic agents. The aim of mechanism design is to create a set of rules such that informed agents find it optimal to truthfully reveal their private information. The mechanism that possesses such a property is called *incentive compatible mechanism*.

In the auction theory each bidder assigns a value to the object being auctioned, and his value is his private information. The value is distributed according to a probability distribution. This value could be regarded as a bidder's reserve price or the maximal amount he is willing to pay for the item. Each bidder determines a bid at the auction that depends on his value. Some auction forms have the property that bidders truthfully reveal their private information, i.e. they bid their true value. Auctions that possess this property could be considered as incentive compatible mechanisms, and in that vein auctions could be regarded as a special case of mechanism design.

There are two possible approaches to auction theory. As we briefly described, the first approach employs mechanism design to analyse auctions. This approach was used by Milgrom (2004) in his book. The other possible approach is to analyse auctions as games of incomplete information and to employ Harshanyi's concept of Bayesian-Nash equilibrium. This approach dominates in Krishna's (2002) book. Menezes and Monteiro (2004) predominantly use the game-theoretic approach, although they give substantial attention to the mechanism design approach. In this paper we will use the game-theoretic approach, which in our opinion is more intuitive.

Auctions are important not only from the theoretical point of view but also from the practical point of view. This field sheds light on the intimate relationship between economic theory and practice, because in this field theory is used to make practical improvements while theoretical findings are confronted with real data in order to test their validity. One of the leading theorists and practitioners in auction design, Paul Klemperer, has written a book (2004) that explains the relationship between auction theory and practice. In his book he talks about the auction designed to sell spectrum rights in the UK, in whose design he participated. A theory is meaningless without empirical testing. A theory can be logically coherent, but can fail when confronted with data. The development of auction theory is accompanied by the development of empirical research of auctions. Paarsch and Hang Hong (2006) give a survey of empirical models used to test auction data.

In this paper we will talk about the four most commonly used auctions. The first two are known as *open auctions* because bidders publicly submit their bids. The most commonly used auction form is the *English auction*, in which the auctioneer starts the auction with a low price and raises that price gradually until only one bidder expresses a willingness to buy at that price. The last bidder is the winner and he gets the object and pays the price at which the previous bidder dropped out. There exists another type of English auction, which is known as Japanese auction in which the price is raised continuously on an electronic display and a bidder presses a button to indicate that he is still active. Once the bidder releases the button he drops out from the auction. The last bidder who stays in wins the auction and pays the price at which the previous bidder dropped out. The other form of open auction is a *Dutch auction,* which is used in Netherlands to sell flowers. The auctioneer starts the auction with a high price and he lowers that price gradually. The first bidder who indicates the interest to buy the item at the price posted by the auctioneer wins the auction and he pays that price.

The other two auction forms are known as sealed-bid auctions, because bidders submit their bids in sealed envelopes. In a *first-price sealed-bid auction*, the bidder who has submitted the highest bid is the winner and he pays his bid. In a *second-price auction*, the bidder who has submitted the highest bid is the winner, but he pays the second highest bid. This auction has the property that each bidder finds it optimal to submit a bid that is equal to his value. In the terminology of the mechanism design, a second-price auction is an incentive compatible mechanism. This property of the second-price auction was first noted by Vickrey[1] (1961) and a similar mechanism is now applied in other areas of economics, and this mechanism is known as the Vickrey-Clarke-Groves mechanism, which is widely used in the analysis of public goods.

Bidders can have *private values* for the object, i.e. a value that a particular bidder assigns to the object is independent of the values of other bidders. Technically, values are independent and distributed according to probability distribution. If a bidder intends to keep the object for himself, a private value environment is a good description of reality. On the other hand, if bidders compete for a right to exploit mineral resources, such as oil, they might have different estimates of the amount of oil in the soil. That estimate is called a signal. Different bidders have different estimates (signals) that are, roughly speaking, correlated, and a value that a particular bidder assigns to the object depends on the other bidders' signals. In this case bidders values are *interdependent*. A special case of interdependent values is called the *common value* model in which bidders have the same value for the object, but have only noisy signals about that value. For example, if a bidder buys the item and plans to resell it in the future, then private values are not a good approximation. In that case a bidder might only have some estimate of the price at which he can resell the object. Common value is a resale price that is the same for all bidders, but is unknown at the time of auction. In this paper we will confine our attention only to the private value case.

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1 A second-price auction is not equivalent to a Vickrey auction. A Vickrey auction is used in a multiple objects auction which will not be analysed in this paper.

Another classification is possible, since there exists a difference between auctions in which there is only a single indivisible object for sale and auctions in which multiple objects are sold simultaneously or sequentially. The first type of auction is known as a single-object auction and the second type as a multiple-object auction. We will study only single object auctions.

Finally, there are two ultimate criteria that an auction mechanism has to achieve. The first is *efficiency,* which means that the object has to be sold to the bidder with the highest value. The other aim is *maximization of the seller's expected revenue* from the sale. Sometimes these two criteria are aligned, but sometimes they are in conflict and an auction designer faces a trade-off. One might wonder why we should be concerned with efficiency. If the object is awarded to a bidder who does not have the highest value it might be resold after the auction to a person who values it the most. Unfortunately, it can be shown that due to transaction costs and bargaining under incomplete information, the resale stage will not result in efficient allocation. When values are private and bidders are symmetric (their values are drawn from the same probability distribution), all four auctions allocate efficiently. In the private values environment, the problem with efficiency might arise in first-price and Dutch auctions when bidders are asymmetric.

As we have announced, in this paper we will be dealing with private value single-object auctions. To the best of our knowledge, this is the first paper in our country that deals with the theoretical aspects of auction design. The rest of the paper is organized as follows. In the second part, we derive equilibrium bidding strategies in a first-price auction, and we will see that each bidder bids lower than his value. In third part, we prove that in a second-price auction there exists equilibrium in the dominant strategies in which bidders bid their values. In the fourth part, we prove that under certain assumptions all four auctions yield the same expected revenue to the seller. The fifth part deals with the optimal reserve price that the seller determines to maximize his revenue. In the sixth part, we will see that auctions can be studied by using standard tools from demand theory, such as marginal revenue. In the seventh part, we will see that revenue equivalence does not hold if one of the assumptions underlying revenue equivalence is relaxed. The eighth part is dedicated to bidding rings, or collusion in auctions, when bidders act cooperatively. Finally, the conclusion follows.

2. FIRST-PRICE AUCTION

Let us assume that each bidder assigns a value V_i (a random variable) to the object that is sold at the auction, and that these values are identically and independently distributed at the interval $V_i \in [0,\omega]$, with the density function $f(\cdot)$ and the distribution function $F(\cdot)$. The density function is obtained by taking the derivative of the distribution function. V_i can be interpreted as a bidder's reserve price, a maximal price that he is willing to pay for the object. Bidder i knows the realization of his value v_i of V_i, but does not know the values of other bidders. He only knows that the values of other bidders are distributed according to $F(\cdot)$. Finally, all bidders are risk neutral, i.e. they maximize expected profits.

The strategy of each bidder is to determine his bid for each possible value. In other words, a bidder's strategy is his bidding function, which is a mapping from the set of values to the set of bids: $b_i : [0,\omega] \to R_+$. We will assume that this function is strictly increasing, but in some papers it is assumed that this function is only non-decreasing.

The rules of the first-price auction are as follows. Each bidder submits to the auctioneer a sealed bid that depends on his evaluation. The bidder who has submitted the highest bid obtains the object and pays his bid. We will show that in a first-price auction each bidder has an incentive to shade his bid, i.e. to bid less than his value. Therefore, a first-price auction is not a truth-revealing mechanism, because bidders do not have an incentive to report their true values.

We will demonstrate the preceding argument informally, before delving into a formal proof. If bidder i wins the item, his payoff is $\Pi_i = v_i - b_i$, otherwise his payoff is 0. It is obvious from the profit function that if the bidder submits a bid equal to his value his payoff would be 0. Therefore, each bidder will bid less than his value in order to obtain a positive profit.

Now we will derive equilibrium bidding strategies, by following Riley and Samuleson (1981). Suppose there are N bidders. We will first show that each bidder bids according to $b_i(v_i)$ and that these strategies constitute a Nash equilibrium. Consider a bidder 1, and suppose that he bids according to $b_1(x)$, where $x \neq v_1$, whereas other bidders bid according to $b_i(v_i)$. Bidder 1 will win the auction if all other N-1 values are less than x. The probability that a particular bidder has a value less than x is $F(x)$, and therefore bidder 1 wins with probability $F(x)^{N-1} \equiv G(x)$.

The expected profit of bidder 1 who has a value v_1 and bids as if his value is x is:

$$\Pi_1(v_1, x) = v_1 \cdot G(x) - P(x), \tag{1}$$

where $P(x)$ is the expected payment bidder 1 has to make to the auctioneer. Without loss of generality, we will suppose that the bidder with the value of 0 has an expected profit of 0, $\Pi_1(0,0) = 0$, which implies that $P(0) = 0$. On the other hand, if bidder 1 follows the strategy $b_1(v_1)$, his expected profit is:

$$\Pi_1(v_1, v_1) = v_1 \cdot G(v_1) - P(v_1). \tag{2}$$

By differentiating (1) with respect to x, we have that:

$$P'(x) = v_1 \cdot \frac{d}{dx} G(x). \tag{3}$$

By integrating the last expression and by using the boundary condition that $P(0) = 0$, we have that:

$$P(v_1) = \int_0^{v_1} y dG(y). \tag{4}$$

Integrating the last expression by parts, we obtain:

$$P(v_1) = v_1 G(v_1) - \int_0^{v_1} G(y) dy. \tag{5}$$

By using the analogy with (5), we obtain that:

$$P(x) = xG(x) - \int_0^{x} G(y) dy. \tag{6}$$

By using (1), (2), (5) and (6), we have that:

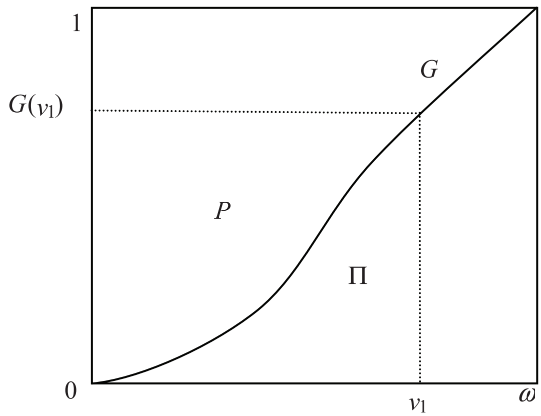
$$\Pi_1(v_1,v_1)-\Pi_1(v_1,x)=v_1\cdot G(v_1)-v_1G(v_1)+\int_0^{v_1}G(y)dy-v_1\cdot G(x)+xG(x)-\int_0^{x}G(y)dy\ , \quad (7)$$

$$\Pi_1(v_1,v_1)-\Pi_1(v_1,x)=(x-v_1)\cdot G(x)+\int_x^{v_1}G(y)dy\geq 0\,, \quad (8)$$

regardless of whether $x \geq v_1$ or $x \leq v_1$. Therefore, we can conclude that bidder 1 cannot do better than to follow the strategy $b_1(v_1)$, because this strategy maximizes his expected profit.

By substituting (5) in (2), it becomes clear that bidder 1's expected profit can be written as $\Pi(v_1,v_1)=\int_0^{v_1}G(y)dy$. Thus, a bidder's expected profit represents the area below the probability distribution $G(\cdot)$ up to his value (figure 1). From (5) we see that the expected payment is equal to the difference between the rectangle area $v_1G(v_1)$ and the area below the distribution function $G(\cdot)$ (expected profit).

Figure 1: *Bidder's expected payment and profit in a first-price auction*



Since bidder 1's expected payment is $P(v_1)=G(v_1)\cdot b(v_1)$, we can obtain the equilibrium strategy in a first-price auction by using (5):

$$b(v_1)=\frac{P(v_1)}{G(v_1)}=v_1-\int_0^{v_1}\frac{G(y)}{G(v_1)}dy\ . \quad (9)$$

From the last expression we see that the bidder *reduces his bid relative to his value* (bid shading), as we argued informally. The degree of bid shading depends on the number of bidders, and *the larger the number of bidders, the lower the bid shading*, since:

$$\frac{G(y)}{G(v_1)} = \left[\frac{F(y)}{F(v_1)}\right]^{N-1}. \tag{10}$$

At the limit, as $N \to \infty$, $b(v_1) \to v_1$, that is, a bidder bids his value. In fact, a bidder faces the following trade off. *If he reduces his bid relative to his value he will pay less if he wins and he obtains a higher profit, but at the same time he reduces the probability of winning the auction*. As the number of bidders increases, the second effect dominates and a bidder is more concerned about the probability of winning and he lowers the amount of bid shading.

In order to illustrate the preceding result, we will suppose that values are uniformly distributed. For uniform distribution, $F(v) = v$ and $G(v) = v^{N-1}$. By using these assumptions in (9) we have that:

$$b(v) = v - \frac{1}{v^{N-1}} \int_0^v y^{N-1} dy = v - \frac{1}{v^{N-1}} \cdot \frac{v^N}{N} = \frac{N-1}{N} \cdot v. \tag{11}$$

Therefore, with uniformly distributed values, the bidding strategy is linearly increasing the function of value, and as N increases the bid gets closer to the value. The bidder's profit is $v - b(v) = v / N$.

The expected revenue of the seller in a first-price auction is the expected value of the highest bid:

$$R^1 = E[\max(b(v_1),...,b(v_n))] = E[b(\max(v_1,...,v_n)]. \tag{12}$$

We have to determine the distribution function of the random variable $\max(v_1,...,v_n)$ in order to calculate this expectation, since the expected value of a continuous random variable is equal to a definite integral of the random variable times its probability density function. This distribution function is the probability that $\max(v_1,...,v_n) \le v$. Maximal value will be smaller than v if each of the *N* values is smaller than v. Since the probability that one value is smaller than v is $F(v)$, the probability that all *N* values are smaller than v is $F(v)^N$ and this is the distribution function of the random variable $\max(v_1,...,v_n)$. Thus, the seller's expected revenue from a first-price auction is:

$$R^1 = \int_0^{\omega} b(v)d\,F(v)^N = \int_0^{\omega} Nb(v)\,F(v)^{N-1} f(v)dv\,. \tag{13}$$

3. SECOND-PRICE AUCTION

In a second-price auction each bidder submits a sealed bid to the auctioneer. The bidder with the highest bid wins the item, but he pays the second highest bid. As we will see below, this auction is a truth-revealing mechanism, since each bidder has an incentive to report his value truthfully. In contrast to a first-price auction, where each bidder pays his bid, in a second-price auction the price paid by the winning bidder is determined by the bid of the other bidder.

In his seminal paper, Vickrey (1961) has shown that a bidder's dominant strategy is to bid his value. In other words, a second-price auction has an equilibrium in dominant strategies. In order to demonstrate this argument, we will use the following proof.

Let us assume that bidder 1 has a value v_1 and denote his bid by b_1. The highest bid of other bidders is $\hat{b}$. Suppose that bidder 1 underbids, that is bids lower than his value $b_1 < v_1$. When $b_1 > \hat{b}$, bidder 1 still wins the object even with this lower bid. However, when $b_1 < \hat{b} < v_1$, he loses the auction he would have won if he had reported truthfully. In this case, he earns a profit of 0 instead of the positive profit $v_1 - \hat{b}$. Thus, this deviation is not profitable.

The same argument applies to overbidding. In this case $b_1 > v_1$. If $b_1 < \hat{b}$, bidder 1 loses the auction and the outcome is the same as if he had bid his value. However, when $b_1 > \hat{b} > v_1$, bidder 1 wins the auction that he would have lost if he had bid his value. In this case, he ends up with negative profit of $v_1 - b_1 < 0$, instead of profit of 0. Therefore, this deviation is not profitable.

Thus, we have shown that in a second-price auction truth-telling is a dominant strategy. Since the same argument applies to each bidder, the equilibrium of the second-price auction is an equilibrium in dominant strategies.

Let us now calculate the expected revenue of the seller in a second-price auction. The seller's expected revenue is the expected value of the second highest value, which is equal to:

$$R^2 = \int_0^{\omega} yd\, F_2(y)dy = \int_0^{\omega} y\, f_2(y)dy\,, \tag{14}$$

where $F_2(y) = F(y)^N + N \cdot F(y)^{N-1}(1 - F(y))$ is the distribution function of the second highest value ($2nd\ highest\ of\ \{v_1, v_2, ..., v_N\} < y$). The first term represents the event that all *N* values are less than *y* and the second term represents the event that *N*-1 values are less then *y* and one value is higher than *y* and there are *N* different ways in which this can happen. This distribution function can be written as $F_2(y) = N \cdot F(y)^{N-1} - (N-1) \cdot F(y)^N$ and its density is:

$$f_2(y) = N(N-1) \cdot (1 - F(y))F(y)^{N-2} f(y)\,. \tag{15}$$

By substituting (15) in (14), the expected revenue for the seller in a second-price auction becomes:

$$R^2 = N(N-1) \cdot \int_0^{\omega} y(1 - F(y))F(y)^{N-2} f(y)dy\,. \tag{16}$$

Vickrey (1961) pointed out that in second-price auctions bidders are not obliged to do difficult mental calculations to find equilibrium bids as in first-price auctions, and second-price auctions are strategically simpler. This property favours a second-price auction, but as we will see below its major weakness is that it is fragile to collusion.

4. REVENUE EQUIVALENCE

The famous revenue equivalence theorem demonstrates that if values are independent and bidders are risk neutral and symmetric, the two auction mechanisms yield the same expected revenue to the seller, $R^1 = R^2$ (the proof of the revenue equivalence by using the game-theoretic approach is given in appendix A). The result depends crucially on these three assumptions, and if one of these is not satisfied, the revenue equivalence no longer holds. But we will postpone the discussion about relaxing some of the assumptions.

As we have proved in appendix A, a first-price auction and a second-price auction generate the same expected revenue to the seller, but we can extend this result to all four auction forms. First, note that the English auction is strategically equivalent to a second-price auction. This is because in the English auction each bidder stays in until the price determined by the auctioneer reaches his value. By further staying in the auction after the price has become higher than the value, a bidder can only make a loss if he wins, so his dominant strategy is to quit the auction when the price reaches his value. But this is the same strategy as in a second-price auction, and these two auction forms are strategically equivalent. The payoff to the winning bidder is the difference between his value and the second highest value, since the next-to-last bidder dropped out when the price reached his value, the same payoff as in a second-price auction. The expected revenue of the seller is equal to the expectation of the second highest value as in a second-price auction, and thus the expected revenue of the seller in these two auction forms is the same.

Furthermore, a bidder in a Dutch auction will bid less than his value, since he pays his bid and his profit is the difference between his value and his bid. Thus, a bidder in a Dutch auction faces the same problem as a bidder in a first-price auction and these two auction forms are strategically equivalent. The expected revenue of the seller in a Dutch auction is equal to the expected value of the highest bid, the same as in a first-price auction.

Therefore, we can conclude that *if values are independent and bidders are risk neutral and symmetric, first-price, second-price, English and Dutch auction generate the same expected revenue to the seller*.

It is important to note that this principle holds only in expected terms, and that for particular realizations of the two highest values (v_1, v_2) one auction form generates higher revenue for the seller. For example, if values are uniformly distributed, then by using the equilibrium strategies derived above we can conclude that if $N v_1/(N-1) > v_2$, then a first-price auction generates higher revenue than a second-price auction. If the reverse inequality holds, a second-price auction generates higher revenue than a first-price auction.

The revenue equivalence can be extended in the following way. In general, it states that any two auction forms that have the same allocation rule and that give the same expected profit to the bidder with the lowest possible value yield the same expected revenue for the seller. This property is useful in deriving equilibrium bidding strategies in some unusual auction forms such as all-pay auctions in which all bidders pay their bids and the bidder with the highest bid wins the object.

In order to prove this general result we will use the mechanism design approach. Denote bidder *i*'s expected profit as:

$$\Pi_i(v) = v \cdot \Pr_i(v) - P_i(v), \tag{17}$$

where $\Pr_i(v)$ is the probability that bidder i wins the auction and $P_i(v)$ is the bidder's expected payment. If bidder i who has a value v behaves as if he has a value v', his expected profit would be:

$$\Pi_i(v, v') = v \cdot \Pr_i(v') - P_i(v'). \tag{18}$$

A bidder who has a value v' has the expected profit:

$$\Pi_i(v') = v' \cdot \Pr_i(v') - P_i(v'). \tag{19}$$

Subtracting (19) from (18) and, rearranging, we obtain:

$$\Pi_i(v, v') = \Pi_i(v') + (v - v') \cdot \Pr_i(v'). \tag{20}$$

In equilibrium, a bidder with value v does not have an incentive to deviate:

$$\Pi_i(v) \geq \Pi_i(v') + (v' - v) \cdot \Pr_i(v'). \tag{21}$$

The same inequality holds for a bidder with value v':

$$\Pi_i(v') \geq \Pi_i(v) + (v - v') \cdot \Pr_i(v) . \qquad (22)$$

If we consider a continuous case where $v' = v + dv$, we obtain from (21) and (22):

$$\Pr_i(v + dv) \geq \frac{\Pi_i(v + dv) - \Pi_i(v)}{dv} \geq \Pr_i(v) . \qquad (23)$$

In the limiting case when $dv \to 0$, we obtain that the slope of the profit function of bidder *i* is:

$$\frac{d\,\Pi_i(v)}{dv} = \Pr_i(v) . \qquad (24)$$

Integrating this expression, we have that:

$$\Pi_i(v) = \Pi_i(0) + \int_0^v \Pr_i(y)dy . \qquad (25)$$

Now consider any two auction mechanisms that give the same payoff to the lowest possible type $\Pi_i(0)$ and that have the same allocation rule, i.e. they give the same probability of winning $\Pr_i(v)$ to every bidder. According to (25), in these two mechanisms, the expected profit of a bidder with value v is the same. Since the expected profit of a bidder with value v is $\Pi_i(v) = v \cdot \Pr_i(v) - P_i(v)$, where $P_i(v)$ is his expected payment, it follows that a bidder with value v has the same expected payment in these two mechanisms. The seller's expected revenue is equal to the sum of expected payments of all bidders, and it follows that in *any two auction mechanisms that give the same expected profit to a bidder with the lowest possible value and that have the same allocation rule, the expected revenue for the seller is the same*.

In particular, English, Dutch, first-price and second-price auctions give zero expected profit to a bidder with the lowest possible value, and the bidder with the highest value always wins the auction if bidders are symmetric. Thus, these four auction forms give the same expected revenue to the seller.

5. RESERVE PRICES AND ENTRY FEES

In some cases it is optimal for the seller to set a reserve price, a price below which he is not willing to sell the item. If there is only one bidder and no reserve price, he can bid 0 for the item and get it for free. On the other hand, if the seller sets too high a reserve price, he runs a risk of not selling the object. Therefore the seller must balance off these two effects and find the optimal reserve price for which the expected gain from selling at a higher price is equal to the expected loss if the item is not sold. We will first find the optimal reserve price for a seller who is facing a single bidder[2], and then we will generalize the result for a multiple bidder case.

Suppose that the seller attaches a value v_0 to the object for his personal use, and that a single bidder has a value v. If he sells the object at the second-price auction at the reserve price r his payoff is $r - v_0$, otherwise his payoff is 0. The probability that he will sell the object at the reserve price is $P(v > r) = 1 - F(r)$ (the probability that a bidder's value is higher than r when he pays r) and the probability of not selling the object is $P(v < r) = F(r)$. Thus, the seller's expected profit is:

$$\Pi_0(r) = [1 - F(r)](r - v_0) . \tag{26}$$

The first order condition yields:

$$\frac{d\,\Pi_0(r)}{dr} = [1 - F(r)] - f(r)(r - v_0) = 0 . \tag{27}$$

By solving this equation we obtain the optimal reserve price:

$$r = v_0 + \frac{1 - F(r)}{f(r)} . \tag{28}$$

From the last expression we see that a seller *sets a higher reserve price than his personal value for the object*. This solution is similar to a monopolist who sets a higher price than his marginal cost. In the auction context, the seller extracts

[2] The proof for a single bidder case is due to Rasmusen (2007).

more surplus if a bidder has a high value and sacrifices a surplus in the case where a bidder has a low value. When the value of the bidder is such that $r > v > v_0$, the seller retains the item, and this is inefficient since the bidder has a higher value. This result is similar to a *deadweight loss due to a monopoly*. Moreover, the inefficiency of monopoly stems from the fact that monopoly produces less than the competitive level and charges a higher price. In the auction context, the seller reduces the probability of selling the object (quantity) by setting a reserve price that is higher than his value (which can be considered as a marginal cost).

We will now generalize the result for the case of several bidders. Surprisingly, the same formula for the optimal reserve price in a second-price auction derived above applies, which implies that the optimal reserve price is independent of the number of bidders.

From the preceding results we have that $R^2 = \int_0^{\omega} y f_2(y)dy$ and:

$$f_2(y) = N(N-1)\cdot(1-F(y))F(y)^{N-2} f(y) = N \cdot (1-F(y)) \cdot g(y)\,, \tag{29}$$

where $g(y) = (N-1)\cdot F(y)^{N-2} f(y)$ is the density of $G(y) \equiv F(y)^{N-1}$. Therefore, we have that:

$$R^2 = \int_0^{\omega} y f_2(y)dy = N \cdot \int_0^{\omega} y \cdot (1-F(y)) \cdot g(y)dy\,. \tag{30}$$

Since the expected revenue of the seller is equal to the sum of expected payments of all N bidders, the *ex ante* expected payment (expected payment for all possible realizations of values) of a particular bidder in a second-price auction is:

$$E[P] = \int_0^{\omega} y \cdot (1-F(y)) \cdot g(y)dy\,. \tag{31}$$

But when the seller posts a reserve price, the *ex ante* expected payment of a bidder becomes:

$$E[P] = r(1-F(r))G(r) + \int_r^{\omega} y \cdot (1-F(y)) \cdot g(y)dy \,, \tag{32}$$

where the first term represents the expected payment when all bidders have values lower than *r* and a particular bidder has a value higher than *r* and pays the reserve price. This event happens with probability $(1-F(r))G(r)$. The second term represents the event where the second-highest value is higher than *r* when the winning bidder pays the second-highest value. The seller retains the object with probability $F(y)^N$ and he attaches a value of v_0 to the object. His expected payoff is:

$$\Pi_0 = N \cdot E[P] + F(y)^N \cdot v_0 \,. \tag{33}$$

Differentiating this expression with respect to *r*, and by using Leibnitz's rule[3] for differentiating integrals we obtain:

$$\frac{d\Pi_0}{dr} = N \cdot [1-F(r)-rf(r)] \cdot G(r) + r \cdot [1-F(r)] \cdot g(r) - r \cdot [1-F(r)] \cdot g(r) + N \cdot F(r)^{N-1} \cdot f(r) \cdot v_0 = 0 \tag{34a}$$

$$\frac{d\Pi_0}{dr} = N \cdot [1-F(r)-rf(r)] \cdot G(r) + N \cdot G(r) \cdot f(r) \cdot v_0 = 0 \,. \tag{34b}$$

$$\frac{d\Pi_0}{dr} = N \cdot \left[1-(r-v_0)\frac{f(r)}{1-F(r)}\right] \cdot (1-F(r))G(r) = 0 \,. \tag{35}$$

From the last result we obtain the optimal reserve price:

$$r = v_0 + \frac{1-F(r)}{f(r)} \,. \tag{36}$$

We have shown that with many bidders the optimal reserve price is the same as in the case of a single bidder. The reason underlying this result is that the reserve price is important only in the case where a highest value is higher than

3 Leibnitz's rule says that
$$\frac{\partial}{\partial r}\int_{a(r)}^{b(r)} f(x,r)dx = f(b(r),r) \cdot \frac{\partial b(r)}{\partial r} - f(a(r),r) \cdot \frac{\partial a(r)}{\partial r} + \int_{a(r)}^{b(r)} \frac{\partial f(x,r)}{\partial r} dx \,.$$

the reserve price and all other values are lower than the reserve price, and this explains why the *optimal reserve price does not depend on the number of bidders*.

Now suppose that there are two bidders and that the seller has a zero value for the object. In finding the optimal reserve price the seller faces the following trade off. If the reserve price is too high, it can happen that the highest value is lower than the reserve price and the object remains unsold. The probability of this event is $F(r)^2$, the loss is *r*, and the expected loss is $rF(r)^2$. But in the case where the highest value is higher than the reserve price and the second highest is lower, the seller's expected gain is *r* and this event happens with probability $2 \cdot F(r) \cdot (1 - F(r))$ (there are two ways in which this can occur). The expected gain is $2 \cdot r \cdot F(r) \cdot (1 - F(r))$. For small *r* the expected gain exceeds the expected loss and the reserve price increases the expected revenue of the seller. But as *r* rises, at some point the expected loss outweighs the expected gain and it is not profitable to increase further the reserve price.

In order to illustrate the trade off, suppose that the value is uniformly distributed in [0,1] and that $v_0 = 0$. In that case $F(r) = r$ and $f(r) = 1$. The expected gain from setting a reserve price is $2 \cdot r^2 \cdot (1 - r)$ and the expected loss is r^3. It is obvious that the expected gain is decreasing in *r* and that the expected loss is increasing in *r* and the optimal reserve price can be found by equating the expected loss with the expected gain:

$$2 \cdot r^2 \cdot (1 - r) = r^3 \Rightarrow r = \frac{1}{2}. \tag{37}$$

This result shows that it is optimal for a seller to exclude some bidders with values lower than the reserve price from the auction and this fact is referred to as the *exclusion principle*. Bidders with lower values are excluded from the auction, which decreases the seller's expected revenue, but bidders who participate bid more aggressively, which increases the seller's expected revenue since an active bidder now has to beat other bidders whose values belong to interval $[r, \omega]$ instead of $[0, \omega]$.

Bulow and Klemperer (1986) point out that it is more beneficial for a seller to have an additional bidder than to post a reserve price in English and second-price auctions. As in the preceding example, value is uniformly distributed in $[0,1]$ and $v_0 = 0$. The optimal reserve price is 1/2, and if there is only one bidder the seller will sell one half of the time, resulting in the expected revenue of 1/4. If there are two bidders, but the seller cannot post a reserve price, his expected revenue would be equal to $E[\min(v_1, v_2)]$. The distribution function of the random variable $\min(v_1, v_2)$ is:

$$F^{II}(v) = 2 \cdot F(v) \cdot (1 - F(v)) + F(v)^2 . \tag{38}$$

The first term represents the probability that one value is higher than v and that the other is lower than v and there are two different ways in which this can occur. The second term represents the probability that both values are less than v. In the case of uniform distribution where $F(y) = y$, the distribution function becomes $F^{II}(y) = 2 \cdot y \cdot (1 - y) + y^2 = 2y - y^2$. The associated density is:

$$f^{II}(y) = d\, F^{II}(y) = 2 - 2y . \tag{39}$$

The expected value of the random variable $\min(v_1, v_2)$ is:

$$E[\min(v_1, v_2)] = \int_0^1 y\, f^{II}(y)dy = \int_0^1 y(2 - 2y)dy = 2\int_0^1 (y - y^2)dy = 2\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{1}{3} . \tag{40}$$

Thus, the expected revenue is higher when there are two bidders and no reserve price, than when there is one bidder and the seller posts a reserve price. *The additional bidder is worth more than the reserve price in English and second-price auctions.*

The seller can exclude the same set of bidders by using an *entry fee*, the amount that each bidder has to pay in order to participate in the auction. The same set of bidders can be excluded by using an entry fee that is equal to the profit of a bidder with value *r*, which means that only bidders with values higher than *r* will find it worthwhile to pay the entry fee. In order to determine the optimal entry fee, note that the expected payment of a bidder in a first-price auction can be written as:

$$P(v) = G(v)b(v) = G(v)v - \int_0^v G(y)dy \,. \tag{41}$$

Since $G(v)v$ is the total expected utility of obtaining the object and $P(v)$ is a payment, it follows that the expected profit is equal to $\Pi = G(v)v - P(v) = \int_0^v G(y)dy$. As we have previously shown, a bidder's expected profit is equal to the area below the distribution function. By using that result, we can conclude that an optimal entry fee that excludes the same set of bidders as a reserve price of *r* is equal to the expected profit of a bidder with value *r*:

$$e = \int_0^r G(y)dy \,. \tag{42}$$

6. MARGINAL REVENUE APPROACH

Bulow and Roberts (1989) have shown that it is possible to study auctions by using standard tools from monopoly pricing, such as marginal revenue curves. The probability that a buyer's value is higher than v, $1 - F(v) \equiv q$ can be regarded as a quantity, because when a seller posts a price of v he will sell with probability $1 - F(v)$. This demand curve gives the maximal price when the quantity is 0 and minimal price when the quantity is 1 (figure 2). From the expression for quantity we can obtain the price:

$$v = F^{-1}(1-q) \,. \tag{43}$$

The total revenue is the product of total quantity and the price:

$$R = F^{-1}(1-q)q \,. \tag{44}$$

From the last expression we can obtain marginal revenue:

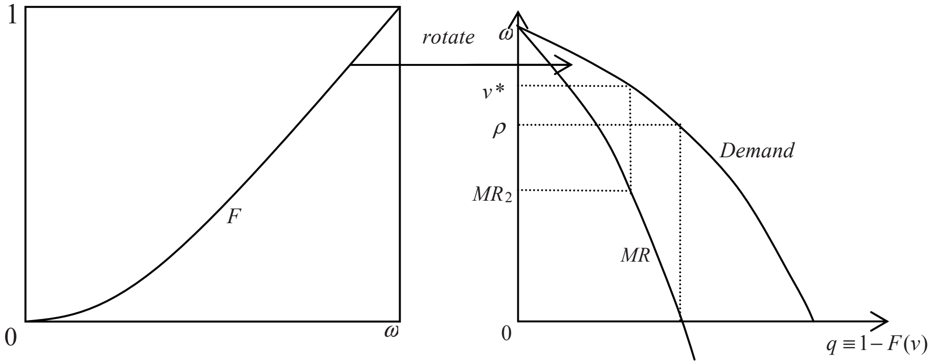
$$MR \equiv \frac{dR}{dq} = F^{-1}(1-q) + q\frac{dF^{-1}(1-q)}{dq} \,. \tag{45}$$

$$MR \equiv \frac{dR}{dq} = F^{-1}(1-q) - \frac{q}{f(F^{-1}(1-q))} \,. \tag{46}$$

By substituting the expression for $v = F^{-1}(1-q)$, we obtain:

$$MR \equiv \frac{dR}{dq} = v - \frac{1-F(v)}{f(v)}. \tag{47}$$

Figure 2: *Probability distribution and marginal revenue*



Graphically, the inverse demand function can be obtained by rotating the distribution function. The marginal revenue curve lies below the inverse demand function.

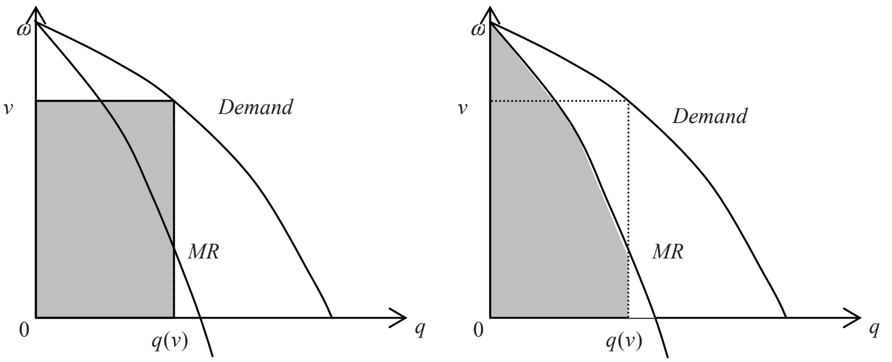
Myerson (1981) shows that in the optimal auction mechanism, which maximizes the seller's expected revenue, the seller should sell the object to a buyer with the highest *virtual valuation* defined as:

$$\psi_i(v_i) \equiv v_i - \frac{1-F_i(v_i)}{f_i(v_i)}. \tag{48}$$

The price paid by the winning buyer would be the lowest possible price that would let him win the auction.

Bulow and Klemperer (1996) prove that the expected revenue of the seller is equal to the expected marginal revenue of the winning bidder by using the marginal revenue approach and justify Myerson's (1981) result that it is optimal to allocate the item to the bidder with the highest marginal revenue. We will explain their results by following their paper as well as Klemperer (2004).

Figure 3a: *Expected Revenue as rectangle* **Figure 3b:** *Expected Revenue as area below MR*



From figure 3a we see that the total expected revenue for the seller when the price is v is equal to $v \cdot q(v) = v \cdot (1 - F(v))$. We see from figure 3b that it is possible to calculate the seller's expected revenue as the area below the marginal revenue curve up to the quantity $q \equiv 1 - F(v)$. Therefore, the last result implies that:

$$v \cdot q(v) = \int_0^{q(v)} MR(v(q))dq\,. \tag{49}$$

Suppose that v is the second highest value and this is the price paid by the winning bidder in an English or second-price auction. Solving the last equation we obtain that the price paid is equal to the expected marginal revenue of the bidder conditional on the bidder's value exceeding v:

$$v = \frac{1}{q(v)} \int_0^{q(v)} MR(v(q))dq\,. \tag{50}$$

A bidder whose value exceeds v is the winning bidder and the seller's expected revenue in an English or a second-price auction is equal to the expected marginal revenue of the winning bidder. Thus, we have proved, at least intuitively, why the optimal auction mechanism maximizes the seller's expected revenue when the seller allocates the object to the bidder with the highest marginal revenue.

Myerson's virtual valuation corresponds to the marginal revenue defined by Bulow and Roberts (1989) and they use the marginal revenue approach to illustrate Myerson's (1981) abstract result. Bulow and Roberts (1989) define the following *second marginal revenue auction* in which the bidder with the highest marginal revenue wins the item and he pays the lowest possible value that would still let him win the auction. The seller is supposed to have the marginal revenue of zero. In order to illustrate how this auction works, we will consider three possible cases. If only bidder 1 has positive marginal revenue and all other bidders have negative marginal revenues, he will win the auction and the price he pays v^* is equal to the value that would make his marginal value equal to zero $MR_1(v^*) = 0$, since this is the lowest possible value that would let him win. For any value lower than this, his marginal revenue would be negative and he would not win the object since the seller has marginal revenue of zero. The graphical illustration of this price could be explained with figure 2. The price paid by the bidder is denoted ρ and this is the lowest possible price he could pay. If some other bidder has positive marginal revenue, the price paid will be higher. The bidder can conclude that it is worthless participating in the auction if his value is lower than ρ. If more than one bidder has a positive marginal revenue, and the second highest marginal revenue is equal to MR_2, the price paid by a bidder who wins the auction, bidder 1 say, is the lowest possible value v^* that would still let him win the auction, i.e. $MR_1(v^*) = MR_2$. By solving this equation we obtain that $v^* = MR_1^{-1}(MR_2)$ (figure 2).

In this auction each bidder reveals *truthfully his marginal revenue*, since the price paid by the winning bidder is independent of his marginal revenue. On the other hand, *this mechanism is not efficient*, since the bidder with the highest marginal revenue need not be the bidder with the highest value.

In order to illustrate the functioning of the second-marginal revenue auction, suppose that there are two bidders. Bidder 1 has a value uniformly distributed in $v_1 \in [0,10]$, whereas bidder 2 has a value uniformly distributed in $v_2 \in [10,30]$.

Bidder 1 can be considered as having the inverse demand function $v_1 = 10 - 10q_1$. Since $q_1 \equiv 1 - F(v_1)$ represents the probability that a bidder's value exceeds v_1, for $v_1 = 0$, $q_1 = 1$ and for $v_1 = 10$, $q_1 = 0$. By the same fashion, the inverse demand curve for bidder 2 could be written as $v_2 = 30 - 20q_2$. We know from monopoly pricing that marginal revenue has a twice-steeper slope than inverse demand function. Thus, for bidder 1 the marginal revenue is $MR_1 = 10 - 20q_1$ and for bidder 2 $MR_2 = 30 - 40q_2$. By substituting the demand functions $q_1 = (10 - v_1)/10$ and $q_2 = (30 - v_2)/20$, we obtain the following marginal revenue functions:

$$MR_1 = 10 - 20 \cdot \left(\frac{10 - v_1}{10}\right) = 2v_1 - 10, \tag{51}$$

$$MR_2 = 30 - 40 \cdot \left(\frac{30 - v_2}{20}\right) = 2v_2 - 30. \tag{52}$$

We will now analyse three cases. We can conclude that $MR_1 \geq 0$ when $v_1 \geq 5$ and $MR_2 \geq 0$ when $v_2 \geq 15$. Bidder 1 wins the object when $v_1 \geq 5$ and $MR_1 \geq MR_2 \Leftrightarrow v_1 > v_2 - 10$ and the price he pays is $\max(5, v_2 - 10)$, since he pays 5 if $MR_2 = 0$ and the lowest possible price with which he can win if $MR_2 \geq 0$ is $v_1 = v_2 - 10$. If $v_2 \geq 15$ and $v_2 > v_1 + 10$, bidder 2 wins and the price he pays is $\max(15, v_1 + 10)$. If $v_1 < 5$ and $v_2 < 15$, the seller retains the object. We will take the following example. If $v_1 = 8$ and $v_2 = 17$, then $MR_1 = 6 \geq MR_2 = 4$. Thus, bidder 1 wins the object even though he has the lower value and the allocation is inefficient. Bidder 1 pays the price such that $MR_1(v^*) = MR_2 = 4$. Solving the equation $2v^* - 10 = 4$, it follows that bidder 1 pays a price equal to 7.

In the preceding example bidder 1 obtains a profit of 1 since his value is 8 and he pays a price of 7. This profit can be referred to as the *informational rent* since the bidder knows his value and the seller must give him a rent to induce him to reveal his private information.

Finally, note that the marginal revenue approach is similar to monopoly pricing, because the bidder with a negative marginal revenue does not obtain the object. This is similar to a monopolist who never sells a quantity such that his marginal revenue is negative. There is one more analogy with monopoly pricing. A monopoly maximizes profit when marginal revenue is equal to marginal cost. In the auction context, marginal cost can be interpreted as a seller's value v_0. Equating marginal revenue with marginal cost we obtain the optimal reserve price derived in the preceding part. If the seller wishes to increase the probability of selling the object (quantity) he has to decrease the reserve price. The same applies to a monopoly that has to decrease the price to sell more.

7. RELAXING THE REVENUE EQUIVALENCE ASSUMPTIONS

We have noted that the revenue equivalence holds only with private values and with risk neutral and symmetric bidders. *If one of these assumptions is relaxed, revenue equivalence does not hold*. In the following discussion we will study the impact of removing one of the three assumptions while retaining the others.

Risk averse bidders

We will suppose that bidders have the same utility function $u(\cdot)$ and that they are risk averse $u''(\cdot) \leq 0$. First, note that the equilibrium behaviour in a second-price auction is unaffected by risk aversion, because it is still a dominant strategy to bid one's value. Therefore, the expected revenue of the seller is the same regardless of whether bidders are risk averse or risk neutral.

However, the equilibrium behaviour in a first-price auction is affected by risk aversion. It can be proved that the more risk averse bidder bids higher than a less risk averse bidder (see appendix B). This result holds if bidder 1 is risk averse and bidder 2 is risk neutral. In that case, risk averse bidder 1 will bid higher than risk neutral bidder 2.

We have mentioned that a bidder faces the trade off in a first-price auction. If he bids lower, he will pay less if he wins, but at the same time he reduces the probability of winning. When a bidder is risk averse, the increase in expected utility from paying less is lower than the decrease in expected utility if he loses. This explains why a risk averse bidder bids higher than a risk neutral bidder. In other words, *the difference in the bid of a risk averse and a risk neutral bidder can be considered as an insurance premium against the possible loss*.

Therefore, the seller's expected revenue in a first-price auction will be higher when bidders are risk averse than when they are risk neutral. The expected revenue in a second-price auction is the same regardless of risk aversion, and since revenue equivalence holds for risk neutral bidders expected revenues are the same in the two auction forms. On the other hand, *when bidders are risk averse a first-price auction gives strictly higher expected revenue to the seller than a second-price auction.*

In a very technical paper, Maskin and Riley (1984) study optimal auction mechanisms for a risk neutral seller who faces risk averse bidders. They prove that it is optimal for a seller to offer insurance against a loss to bidders with high values to induce them to bid higher than they would bid without the insurance.

Asymmetric bidders

When bidders are asymmetric, they have different probability distributions for their values or have the same probability distribution but with different support. For example, it might be the case that the distribution function of the second bidder dominates the distribution function of the first bidder according to first order stochastic dominance, $F_2(v) \leq F_1(v)$. The other possibility is that the two distribution functions are the same, but that the first distribution function has a support $[0, \omega_1]$ and the second $[0, \omega_2]$, where $\omega_2 > \omega_1$. Finally, it is possible that one distribution dominates the other and has a wider support. Whatever the case, we will call bidder 1 weak bidder and bidder 2 strong bidder.

Vickrey (1961) studied first-price auctions with asymmetric bidders and he reached the conclusion that it is impossible to derive closed form expressions for equilibrium bidding strategies in the general case and he concentrated his attention to one particular case where bidder 1 has fixed value and bidder 2 has a value drawn from some support.

One tractable case where equilibrium bidding strategies in first-price auctions could be derived after solving a system of differential equations is the case of uniform distribution, where the value of the weak bidder belongs to $[0, \omega_1]$ and the value of the strong bidder belongs to $[0, \omega_2]$, where $\omega_2 > \omega_1$. As we will see below, Maskin and Riley (2000) call this type of asymmetry a distribution stretch. In this case it can be shown that a weak bidder bids more aggressively than a strong bidder. This means that if two bidders have the same value the weak bidder will submit a higher bid. But this result shows that with positive probability the allocation in a first-price auction with asymmetric bidders could be inefficient.

On the other hand, a second-price auction is again robust to relaxing one of the assumptions. Regardless whether bidders are symmetric or not, the dominant strategy for each bidder is to bid his value. Thus, while a second-price auction allocates efficiently, a first-price auction allocates inefficiently with positive probability. Since the general revenue equivalence principle implies that the two auction forms that have the same allocation rule give the same expected revenue to the seller, it follows that with asymmetric bidders first and second-price auctions will lead to different expected revenues for the seller. Thus, a first-price auction favours the weak bidder who can win the auction even though he has a lower value. A second-price auction favours the strong bidder because it allocates efficiently.

Maskin and Riley (2000) study different types of asymmetries and find that different asymmetries lead to different rankings of auction forms in terms of a seller's expected revenue. The first case they study is when one bidder's distribution is *shifted to the right*. For example, bidder 1's value belongs to interval $[0,1]$ and bidder 2's value belongs to interval $[1,2]$ and the values are uniformly distributed. Each bidder knows the support of the distribution function of the other bidder. In a first-price auction bidder 2 would therefore always bid 1 because this guarantees that he would win the auction for sure. On the other hand, in a second-price auction, he would pay the second highest value. In other words, he would pay the expected value of the other bidder, which is equal to 1/2. Thus, *when the distribution of the strong bidder is shifted to the right, a first-price auction will generate higher expected revenue for the seller than a second-price auction*.

The other form of asymmetry studied by Maskin and Riley (2000) is a *distribution stretch*, which means that a strong bidder's value is stretched over a wider interval. For example, a weak bidder's value is uniformly distributed over the interval $[0,1]$ and a strong bidder's value is uniformly distributed over the interval $[0,2]$. Maskin and Riley (2000) prove that when a strong bidder in such a case is faced with a weak bidder in a first-price auction, he will bid less aggressively than he would bid if he were to face another strong bidder. On the other hand, a weak bidder would bid more aggressively in a first-price auction than he would bid if he were to face another weak bidder. *The net effect is such that the seller's expected revenue is higher in a first-price auction than in a second-price auction.*

The third type of asymmetry studied by Maskin and Riley (2000) is a *shift of probability mass* to the lower end point of the distribution. For example, suppose that two bidders have degenerate distributions over the interval $[0,2]$ in which all probability is concentrated in 2. If we shift half of the mass to the point 0 for bidder 1, he becomes the weak bidder who has a value 0 or 2 with equal probability. The second bidder becomes the strong bidder who always has a value of 2. In a second-price auction the seller's revenue is positive only if the weak bidder has a value of 2. Therefore, the seller's expected revenue in a second-price auction is equal to $2 \cdot \Pr[v_2 = 2] = 1$, where 2 is the price paid by the winning bidder and $\Pr[v_2 = 2]$ is the probability that the second bidder has a value of 2. In a first-price auction, the strong bidder would bid slightly above zero and he would win with probability 1/2 whenever the weak bidder has a value of 0. His expected payoff from this bid is $0{,}5 \cdot (2-0) = 1$. The strong bidder would never bid more than 1 in equilibrium since he can only reduce his expected payoff. The weak bidder would win for sure with a bid $1+\varepsilon$, for small ε and his *ex ante* expected payoff would be $0{,}5 \cdot (2-(1+\varepsilon)) \approx 0{,}5$. The total expected payoff of the two bidders is 1,5. The bidder who has a value of 2 always wins and the social surplus is equal to 2. The seller's expected revenue is the difference between the social surplus and the sum of the bidder's expected payoffs and is equal to 1/2. Thus for a *shift of probability mass a second-price auction out-performs a first-price auction in terms of the seller's expected revenue.*

Interdependent values

When values are interdependent, the value of one bidder depends on the values of other bidders. In particular, it is assumed that each bidder receives a noisy signal s^i that represents his private estimate of the value of the object. A value that a bidder assigns to the object depends on his signal and on the signals of other bidders. If there are N bidders, bidder i assigns a value

$V_i = v_i(S^1, S^2, ..., S^N)$ to the object, where v_i is his valuation function that is non-decreasing in all N arguments. When values are interdependent, the revenue equivalence fails.

Milgrom and Weber (1982) studied auctions with interdependent values. They introduced the concept of *affiliation,* which is a stronger concept than correlation. Roughly speaking, affiliation means that if one bidder's signal is high it is more likely that others' signals are high rather than low.

In the case of interdependent values, an English auction is no longer strategically equivalent to a second-price auction, for the following reason. When one bidder drops out from an English auction, the other bidders can infer his signal and update their estimates of the value of the object. In a second-price sealed-bid auction such signal extraction is not possible. Since signals are affiliated, bidders will assign higher value to the object in an English auction than in a second-price auction, and Milgrom and Weber (1982) prove that an English auction yields higher expected revenue for the seller than a second-price auction. They are equivalent only when there are two bidders, because when one bidder drops out, the auction is over and the other bidder cannot benefit from inferring the signal of the losing bidder. Milgrom and Weber (1982) also show that a second-price auction yields higher expected revenue than a first-price auction. Thus, the *English auction out-performs a second-price auction and a second-price auction out-performs a first-price auction. This result is known as the revenue ranking or linkage principle.*

This argument was used by Milgrom (1989) to explain the popularity of the English auction. If values are private it yields the same expected revenue to the seller as other auction forms, but when values are interdependent it is revenue superior. Furthermore, it leads to efficient outcomes even with asymmetric bidders and bidders face a simpler decision problem than in first-price and Dutch auctions, as we have mentioned. Its main disadvantages are that it requires the actual presence of bidders at auction, and, as we will soon see, it is fragile to collusive behaviour.

It is important to note that even with affiliated values Dutch and first-price auctions remain strategically equivalent, because the Dutch auction is over when one bidder accepts the price and other bidders cannot benefit from inferring his signal. The same holds for a first-price sealed-bid auction, where a bidder cannot obtain any information about the other bidders' signals. Therefore, we can say that *English and second-price auctions are equivalent in the weak sense,* i.e. they are equivalent only when values are private or there are only two bidders. On the other hand, *Dutch and first-price auctions are equivalent in the strong sense*, i.e. they are equivalent with both private and interdependent values.

8. BIDDING RINGS

Up to this point we have assumed that bidders act non-cooperatively. But bidders can coordinate their activities and make a cartel or a ring in order to increase their profit. In this case there is only one bidder who bids in the name of the ring, while the other bidders do not submit bids. Members of the ring share the profit of the cooperative behaviour. Some auction mechanisms are more susceptible to collusive behaviour. We will show that a second-price auction is vulnerable to collusive behaviour, and that a cartel is not stable in a first-price auction. The study of collusive behaviour is important, because cartels are present in many auctions, and in most cases the victim of collusion is a government agency, as reported by Hendricks and Porter (1989).

Collusion in second-price auctions

Graham and Marshall (1987) analyse cartels at second-price auctions. They describe cartels in the following way. There is a sole bidder who bids in the name of the ring and the profit from the cooperative behaviour is shared among the ring members. Rings have open membership policies, meaning that non-members are invited to join. A ring tries to hide its existence from the auctioneer and the only instrument the auctioneer has when facing a ring is to post a reserve price.

Suppose that there are N bidders at the auction, and that K bidders form a ring whereas other N-K bidders act non-cooperatively. In a second-price auction the price a bidder pays is equal to the second highest value. As we have mentioned, there is only one bidder who bids in the name of the ring and the ring uses a mechanism, which will be explained later, to choose a bidder with the highest value who will represent the ring. *The profit of the ring stems from the fact that the second highest value belongs to a member of the ring who will not submit a bid and the winner pays a lower price than he would pay if the ring did not exist*.

The *ring centre* coordinates the activities of the ring and acts as a banker. The ring centre makes a fixed payment *P* to each member of the ring. Each member of the ring submits a sealed bid to the ring centre, who determines the highest and second highest value, and the bidder with the highest bid will represent the ring at the main auction. This auction, which selects the ring's nominee for the main auction, is called a second-price pre-auction knockout (PAKT). If the ring's representative wins the item at the main auction, he pays the second highest value of all submitted bids to the auctioneer and pays to the ring centre the difference between the second highest value determined in PAKT and the second highest value at the main auction, if this difference is positive. This amount is a ring's profit and we will denote it by δ. This value cannot be negative because the winner is not obliged to pay to the centre if the second highest value determined in the PAKT is lower than the second highest value at the main auction. The ring centre has to make fixed payments prior to the main auction and obtains transfers only when the ring's representative wins the item and obtains a profit, when $\delta > 0$. If there are *K* members of the ring, the ring centre has to pay *KP* to the members of the ring, and the ring's expected profit is $E(\delta)$. *In order to balance the budget in expected terms*, the ring centre will pay $P = E(\delta)/K$ to each ring member. If the ring does not obtain the item, the ring centre will have to pay *KP* and receives nothing. On the other hand, if δ is very large, the ring centre runs a surplus since its revenues exceed its payments. Thus, the ring centre balances the budget in expected terms, but in some cases runs a deficit. In other words, *the budget of the ring centre is not balanced ex post*. This is the main weakness of a PAKT, since it needs an outside banker to finance its workings.

We will now show that the PAKT is an *incentive compatible and individually rational* mechanism. Incentive compatibility means that each member of the ring will report truthfully his value to the centre, and individual rationality means that each member of the ring has an incentive to become a member of the ring.

If the winning bidder is a member of the ring, he pays the second highest value at the main auction and pays the difference between the ring's second highest value and the second highest value at the main auction to the ring centre, provided this difference is positive. In fact the winner pays the second highest of all N values as in an ordinary second-price auction and he reports truthfully his value to the ring centre.

If the ring does not obtain the item, each member benefits since he receives P and if he had acted individually he would have obtained nothing. The same argument applies if the ring obtains the item, but it is awarded to the other member of the ring. The membership is again beneficial, since the member receives P and he would receive nothing by acting individually. Finally, if the ring obtains the item the bidder who is awarded the item benefits from membership since he gets the object and pays the same price he would pay if he acted individually, but receives an additional amount P. Thus, PAKT is an individually rational mechanism.

The cartel does not exert externalities to bidders who are not members of the ring. A non-member will win the item with the same probability as in the case without the cartel. The price he would pay upon winning is the same as without the ring and his expected payment and expected profit are the same. Therefore, the profit of the cartel is equal to the loss of the seller. Finally, the cartel's profit rises with the size of the ring. If we add one more bidder to the ring, the price paid at the main auction by the ring could be lower if that bidder is the one who has the second highest value at the main auction. The cartel is self-enforcing and in Nash equilibrium all bidders are members of the cartel (K=N). Without a reserve price, this cartel could obtain the item for a price of zero and the seller acts strategically and posts a reserve price.

Graham and Marshall (1987) show that the *optimal reserve price increases according to the size of the ring* (for a formal proof of this statement see appendix C). Thus, the seller faces the following trade off. As the size of the ring increases for a given reserve price, the expected revenue of the seller falls, since we have seen that the expected profit of the ring increases with its size and the profit of the ring equals the expected loss of the seller. The seller offsets this effect by raising the reserve price, but at the same time, he increases the probability of retaining the item. Finally, in Nash equilibrium the cartel will be all-inclusive ($K=N$) and the seller will post a reserve price for that size of the ring. In other words, the cartel will offer a "take it or leave it" reserve price to a single bidder in Nash equilibrium.

Collusion in first-price auctions

McAfee and McMillan (1992) analyse all-inclusive cartels at first-price auctions ($K=N$). The reason for studying an all-inclusive cartel is that even if bidders are *ex ante* symmetric, there exists an *ex post* asymmetry in bidding strategies between members and non-members of the cartel. This problem does not appear in second-price auctions. They identify four types of cartel: (i) *tacit* mechanism in which there are no transfers between members and each member's bid depends only on his value; (ii) *coordinative* mechanism in which there are no transfers but a bid submitted by a particular member depends on his value and the values of other members; (iii) *transfer* mechanism in which there are transfers and the budget is balanced ex post, i.e. the budget of the ring centre is balanced for every possible realization of values; (iv) *budget-breaking* mechanism in which the budget is balanced only in expected terms. In further analysis they group the four types into only two types, which differ by the presence or absence of transfers. Cartels that do not make transfer payments are called *weak cartels* and cartels that make side payments are called *strong cartels*.

Weak cartels do not make transfers because of the fear of prosecution by the antitrust authorities. This argument can be supported by the data, because most of the bid-rigging convictions in the US began when one member of the cartel was unhappy with the division of cartel profits and started working against his fellows. McAfee and McMillan (1992) show that *for a weak cartel it is optimal to every member who has a higher value than the reserve price to bid the reserve price*. Since all bidders submit the same bid, the seller allocates the object randomly and the allocation will be inefficient because there is no guarantee that the bidder with the highest value will win the item. Weak cartels sometimes use *rotating bids*. In that case there is a mechanism that designs the order in which members bid. The first bidder on the list is asked whether he is willing to pay the reserve price. If he refuses, the next bidder is asked, and so on. Weak cartels use rotating bids for two reasons. First, identical bids result in the equal probability of obtaining the object for every cartel member, and sometimes cartels wish a different division of cartel profits. Second, the seller can disrupt the agreement by refusing to randomize. If he receives identical bids, he can employ a rule that will allocate the object to the smallest firm or the largest firm. The absence of randomization can destroy the cartel agreement, and a weak cartel could prefer rotating bids.

A strong cartel makes side payments between its members. The strong cartel needs a mechanism to elect a cartel representative. As in a second-price auction, a first-price PAKT can be used. In the pre-auction phase, each bidder submits a bid b^{PAKT} that represents the offer to pay all the members that amount. To simplify the discussion, suppose that the reserve price is equal to zero and that values are uniformly distributed over $[0,1]$. We have derived an equilibrium bidding strategy for this case in a first-price auction $b(v) = [(N-1)/N]\cdot v$.

We argue that in a first-price PAKT symmetric equilibrium strategies are:

$$b^{PAKT} = \frac{b(v)}{N} = \frac{N-1}{N^2}\cdot v\,. \tag{53}$$

If a bidder acts individually he would obtain a profit of $v - b(v)$, and because of the presence of the cartel, the cartel representative will earn an additional profit of $b(v)$, because he will bid 0 and pay that price, since the cartel is all-inclusive. Thus, the cartel's profit is equal to $b(v)$ and is divided equally among its members. The bidder with the highest b^{PAKT} will represent the cartel at the auction.

We will prove that a first-price PAKT is an *individually rational* mechanism. If bidders act non-cooperatively, the winning bidder receives a profit:

$$\pi^{NC} = \left[1 - \frac{N-1}{N}\right]\cdot v = \frac{v}{N}\,. \tag{54}$$

If that bidder is the cartel's representative he will win the auction for sure, and his profit after paying b^{PAKT} to his N-1 fellows is:

$$\pi^C = v-(N-1)\cdot b^{PAKT} = \left[1-\frac{(N-1)^2}{N^2}\right]\cdot v = \left[\frac{N^2-N^2+2N-1}{N^2}\right]\cdot v = \left[\frac{2N-1}{N^2}\right]\cdot v\,. \quad (55)$$

Thus, the cooperative profit of the winning bidder is strictly larger than his non-cooperative profit if:

$$\frac{2N-1}{N^2} > \frac{1}{N}\,, \quad (56)$$

which always holds for $N>1$. The bidders who would lose the auction would receive nothing without the agreement, and receive b^{PAKT} if they participate in the ring. Thus, any bidder cannot do any better than to participate, and a first-price PAKT is an *individually rational* mechanism. It can be shown that in a first-price PAKT each bidder will bid b^{PAKT} according to his true value, which means that a bidder with the highest value will win the PAKT, but the proof is more involved. Finally, a first-price PAKT is not an *incentive compatible* mechanism, since bidders do not reveal their true values.

The seller can respond in four ways to the cartel agreement. First, he can post a reserve price, and that reserve price will be higher than the reserve price in a non-cooperative auction. A bidder's profit could be higher in a non-cooperative auction with a lower reserve price than in a cooperative auction with higher reserve price and this could deter collusion. Second, he can keep the reserve price secret. If bidders do not know the reserve price, they must communicate to determine the bid. But communication increases the risk that a cartel will be prosecuted by the antitrust authorities. Third, the seller can influence the mechanism for determining the cartel's representative, which would make the cartel more fragile. This will be further discussed in the following paragraph. Hendriks and Porter (1989) mentioned another strategy that a seller could use. He could announce the identity of the winning bidder and not his bid and losing bids. This could possibly prevent a weak cartel from submitting identical bids.

Stability of the cartel agreement

Robinson (1985) makes a simple but important contribution to the theory of collusive behaviour. He argues that a cartel agreement is less stable in a first-price than a second-price auction. In order to explain his argument, we will employ the preceding example with uniform distribution. In this example the cartel representative bids 0 and all other bidders do not submit bids in a first-price auction. If all bidders behave in this way a particular bidder who is not elected as the cartel's representative has an incentive to cheat and bid slightly higher than zero and he obtains a higher profit than in cooperation, provided that other cartel members respect the agreement. On the other hand, the cartel in a second-price auction is stable, since each member who has not won the PAKT has no incentive to cheat because he cannot win the main auction at the price that is profitable since there will be at least one bidder with higher value (the cartel's representative) at the main auction. Thus, a *cartel in a first-price auction is less stable than in a second-price auction*. Therefore, from the point of view of the seller who faces a threat of collusive behaviour, a first-price auction might be preferable. The cartel in a first-price auction can be stable if the game is repeated infinitely many times, so cheating will be deterred by the threat of expulsion from future auctions.

9. CONCLUDING REMARKS

The approach in this paper was mainly theoretic, although auctions are important from the practical point of view as well. Klemperer (2004) describes the use of the hybrid Anglo-Dutch auction which was used to sell 3G mobile-phone licenses in UK in 2000. The government intended to sell 4 licenses. In the first part of the auction an English auction was used until 5 bidders remained. In the second stage the 5 remaining bidders submitted sealed bids and bidders with the four highest bids obtained the licenses. The main concerns in choosing this auction design were to attract entry and to deter collusion.

The above story tells us how theoretical achievements can be used in practice. But at the same time a practical auction designer should be cautious, because theoretical results are based on some restrictive assumptions and an auction designer in practice has to identify the most important problems in a particular auction. As we know from game theory, games with incomplete information are particularly fragile to changing the assumptions.

In this paper we have analysed single object auctions with private values and this is the simplest class of auction mechanism. Auctions with interdependent values, and especially multiple-object auctions, are more challenging from the modelling point of view and these topics remain for further research.

APPENDIX

A. Proof of the revenue equivalence by using the game-theoretic approach[4]

Recall that the seller's expected revenue in a first-price auction is:

$$R^1 = \int_0^{\omega} b(v) d\, F(v)^N = \int_0^{\omega} Nb(v)\, F(v)^{N-1} f(v) dv\ . \qquad \text{(A1)}$$

We will integrate this expression by parts:

$$R^1 = \int_0^{\omega} Nf(v)[b(v)\, F(v)^{N-1}] dv = NF(v) b(v)\, F(v)^{N-1} |_0^{\omega} - \int_0^{\omega} NF(v)[b(v)\, F(v)^{N-1}]' dv\ . \qquad \text{(A2)}$$

We have to determine the derivative $[b(v)\, F(v)^{N-1}]'$, but first of all we need a derivative of :

$$b(v) = v - \int_0^{v} \frac{F(y)^{N-1}}{F(v)^{N-1}} dy\ , \qquad \text{(A3)}$$

and we will use Leibnitz's rule which says that:

$$\frac{\partial}{\partial r} \int_{a(r)}^{b(r)} f(x,r) dx = f(b(r),r) \cdot \frac{\partial b(r)}{\partial r} - f(a(r),r) \cdot \frac{\partial a(r)}{\partial r} + \int_{a(r)}^{b(r)} \frac{\partial f(x,r)}{\partial r} dx\ , \qquad \text{(A4)}$$

to find the derivative of the integral. Thus, the derivative of (A3) is:

$$b'(v) = 1 - \frac{[F(v)^{N-1}]^2 - (N-1)\, F(v)^{N-2} f(v) \int_0^{v} F(y)^{N-1} dy}{[F(v)^{N-1}]^2} = (N-1) \cdot \frac{\int_0^{v} F(y)^{N-1} dy}{F(v)^{N}} f(v) \qquad \text{(A5)}$$

[4] The proof relies on Menezes and Monteiro (2004).

$$[b(v)F(v)^{N-1}]' = b'(v)F(v)^{N-1} + b(v)(N-1)F(v)^{N-2}f(v) =$$

$$= (N-1)\cdot\frac{\int_0^v F(y)^{N-1}dy}{F(v)}f(v) + \left[v - \frac{\int_0^v F(y)^{N-1}dy}{F(v)^{N-1}}\right](N-1)F(v)^{N-2}f(v) = v(N-1)F(v)^{N-2}f(v)\cdot \quad \text{(A6)}$$

Substituting this result in (A2) we obtain that:

$$R^1 = Nb(\omega) - \int_0^\omega N(N-1)vF(v)^{N-1}f(v)dv\,. \quad \text{(A7)}$$

We know that equilibrium bidding strategy in a first-price auction is:

$$b(v) = v - \int_0^v \frac{G(y)}{G(v)}dy\,, \quad \text{(A8)}$$

where $G(y) \equiv F(y)^{N-1}$.

Now, we have that:

$$F(v)^{N-1}b(v) = vF(v)^{N-1} - \int_0^v F(y)^{N-1}dy\,. \quad \text{(A9)}$$

Integrating the second term by parts we obtain:

$$F(v)^{N-1}b(v) = vF(v)^{N-1} - yF(y)^{N-1}|_0^v + \int_0^v y(N-1)F(y)^{N-2}f(y)dy = \int_0^v y(N-1)F(y)^{N-2}f(y)dy\,. \quad \text{(A10)}$$

From (A10) it follows that:

$$b(\omega) = \frac{\int_0^\omega v(N-1)F(v)^{N-2}f(v)dv}{F(\omega)^{N-1}} = \int_0^\omega v(N-1)F(v)^{N-2}f(v)dv\,. \quad \text{(A11)}$$

By substituting this result in (A7) we have that:

$$R^1 = \int_0^\omega vN(N-1)F(v)^{N-2}f(v)dv - \int_0^\omega N(N-1)vF(v)^{N-1}f(v)dv =$$

$$= N(N-1)\cdot\int_0^\omega vF(v)^{N-2}f(v)[1-F(v)]dv = R^2\,, \quad \text{(A12)}$$

where we have derived expected revenue in a second-price auction, R^2, in (16) in the text. ▪

B. Proof that equilibrium bidding strategy is increasing in bidder's risk aversion in a first-price auction[5]

A risk neutral bidder maximizes expected profit, whereas a risk averse bidder maximizes expected utility. Bidder 1's expected utility is:

$$\Pi_1 = F^{N-1}(v) \cdot u(v - b(v)) . \tag{B1}$$

In equilibrium a bidder bids according to his true value. The first order condition gives:

$$(N-1) \cdot F^{N-2}(v) f(v) \cdot u(v - b(v)) - F^{N-1}(v) \cdot u'(v - b(v)) b'(v) = 0 . \tag{B2}$$

Rearranging terms, we obtain the following differential equation:

$$b'(v) = (N-1) \frac{f(v)}{F(v)} \frac{u(v - b(v))}{u'(v - b(v))} . \tag{B3}$$

We will use the boundary condition that $b(0) = 0$ and suppose that bidder 2 has a higher coefficient of absolute risk aversion:

$$A_2 = -\frac{u_2''(v)}{u_2'(v)} > A_1 = -\frac{u_1''(v)}{u_1'(v)} \geq 0 . \tag{B4}$$

We have to prove that the more risk averse bidder 2 bids higher than the less risk averse bidder 1, $b_2(v) > b_1(v)$.

From (B3) it follows that if:

$$\phi(v) = \frac{u_2(v)}{u_2'(v)} - \frac{u_1(v)}{u_1'(v)} > 0 , \tag{B5}$$

[5] This proof closely follows Riley and Samuelson (1981).

for all $v > 0$, then $b_2'(v) > b_1'(v)$ and $b_2(v) > b_1(v)$ because of the boundary condition $b(0) = 0$. Since $u(\cdot)$ is strictly increasing and $u(0) = 0$, we have that for all $v > 0$:

$$\frac{u(v)}{u'(v)} > \frac{u(0)}{u'(0)} = 0\,. \tag{B6}$$

From (B5) we see that for all v such that $\phi(v) = 0$ or $\frac{u_2(v)}{u_2'(v)} = \frac{u_1(v)}{u_1'(v)}$, $\phi(v)$ is strictly increasing, since the following inequality holds for $A_2 > A_1 \geq 0$:

$$\phi'(v) = \left(-\frac{u_2''(v)}{u_2'(v)}\right)\left(\frac{u_2(v)}{u_2'(v)}\right) - \left(-\frac{u_1''(v)}{u_1'(v)}\right)\left(\frac{u_1(v)}{u_1'(v)}\right) = A_2\left(\frac{u_2(v)}{u_2'(v)}\right) - A_1\left(\frac{u_1(v)}{u_1'(v)}\right) > 0\,. \tag{B7}$$

Thus, using the boundary condition that $\phi(v) = 0$ and the fact that $\phi'(v) > 0$, it follows that $\phi(v) > 0$. Since $\phi(v) > 0$, we have proved that $b_2(v) > b_1(v)$. ■

C. Proof that the equilibrium reserve price increases according to the size of the ring[6]

Suppose that the ring consists of *K* bidders. Denote by Y_1^K the highest value in the ring and denote by Z_2^K the second highest bid at the main auction. If the seller posts a reserve price *r*, the price paid at the main auction will be Z_2^K if $Z_2^K > r$, otherwise the price will be *r*. Denote by $F^K(\cdot)$ the distribution function of Z_2^K and by $f^K(\cdot)$ the corresponding density. As usual, $G(\cdot)$ is the distribution function of the highest value and $g(\cdot)$ is the corresponding density. The expected selling price for the seller is:

$$r \cdot (F^K(r) - G(r)) + \int_r^{\omega} z\, f^K(z)dz\,. \tag{C1}$$

The first term represents the event that the item is sold at the reserve price. This occurs when the highest value is higher than the reserve price and the second highest is lower than the reserve price. The integral represents the event that the first and second highest values are higher than *r* when the winner pays the second highest value Z_2^K.

6 The proof is based on Krishna (2002).

The seller maximizes his revenue when the derivative of (C1) is equal to zero:

$$F^K(r) - G(r) + r \cdot f^K(r) - r \cdot g(r) - r \cdot f^K(r) = 0\,, \tag{C2}$$

$$F^K(r^*) - G(r^*) - r^* \cdot g(r^*) = 0\,, \tag{C3}$$

where we have used Leibnitz's rule to find the derivative of the integral. The optimal reserve price r^* solves the last equation.

Now, suppose that an additional bidder joins the cartel and that the cartel has $K+1$ members. Denote by Z_2^{K+1} the second highest value at the main auction in this case and denote by $F^{K+1}(\cdot)$ the distribution function of Z_2^{K+1} and by $f^{K+1}(\cdot)$ the corresponding density. Z_2^{K+1} differs from Z_2^K in only two cases. If the bidder K+1 who joins the cartel had the second highest value at the main auction, the second highest value will be smaller once he joins the cartel $Z_2^{K+1} < Z_2^K$. In the second case, if bidder K+1 won the main auction and cartel had the second-highest value, the new second highest value will be again smaller than the old one $Z_2^{K+1} < Z_2^K$. In all other cases the second highest value will remain unchanged $Z_2^{K+1} = Z_2^K$. We have shown that with positive probability the second highest value at the main auction will be lower when an additional bidder joins the cartel ($Z_2^{K+1} \leq Z_2^K$) and this implies that $F^K(\cdot)$ dominates $F^{K+1}(\cdot)$ by first order stochastic dominance. First order stochastic dominance implies that if a seller prefers $F^K(\cdot)$ to $F^{K+1}(\cdot)$, then $F^K(\cdot) \leq F^{K+1}(\cdot)$. The expected selling price for the seller when the cartel is of size K+1 is:

$$r \cdot (F^{K+1}(r) - G(r)) + \int_r^{\omega} z\, f^{K+1}(z)dz\,. \tag{C4}$$

The derivative of (C4) at the level of the previously optimal reserve price is r^* is positive because $F^K(\cdot) \leq F^{K+1}(\cdot)$:

$$F^{K+1}(r^*) - G(r^*) - r^* \cdot g(r^*) \geq 0\,. \tag{C5}$$

The last inequality implies that the optimal reserve price for a ring of size K+1, r^{**}, must be higher than the previously optimal reserve price r^*. ■

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