THE DUAL ROLE OF EQUILIBRIUM PRICE IN COMPETITIVE ECONOMIES WITH ASYMMETRIC INFORMATION

ABSTRACT: This paper analyses equilibrium in competitive markets with asymmetrically informed agents. In contrast to Walrasian equilibrium, where equilibrium price is only an indicator of relative scarcity, in the models studied in this paper equilibrium price has two additional roles. It conveys and aggregates the private information of agents in the economy. Each agent infers the private information of other agents by studying the equilibrium price. This implies that agents in this setting have higher cognitive capabilities than Walrasian agents. The equilibrium concept used to describe these additional roles of equilibrium price is called Rational Expectations Equilibrium (REE).

KEY WORDS: Fully revealing rational expectations equilibrium; Noisy rational expectations equilibrium; Information aggregation; Stock market crashes.

JEL CLASSIFICATION: D82, G14.
1. Introduction

In this paper we will analyse a concept of competitive equilibrium called rational expectations equilibrium. In the Walrasian competitive equilibrium framework, economic agents do not infer any private information from the equilibrium price and the equilibrium price is an indicator of the relative scarcity of some commodity. In contrast to the Walrasian paradigm, the equilibrium price plays a dual role in the rational expectations equilibrium: it conveys information about the relative scarcity of some commodity, but also conveys and aggregates private information contained in the economy when agents are asymmetrically informed (Grossman, 1981; 1989). This idea of the dual role of the equilibrium price stems from Hayek (1945) but economists were able to formalize this idea only after the rational expectations equilibrium framework had been developed.

Let us postulate that an economy consists of $I$ asymmetrically informed agents, and that agent $i$ possesses information set $(i^{inf})^i$. In the Walrasian framework, the demand function$^1$ of agent $i$ would be $X_i(p,(i^{inf})^i)$, and the equilibrium price would incorporate all private information $p((i^{inf})^1,...,(i^{inf})^I)$. Nevertheless, a single economic agent does not try to infer private information of other agents from the equilibrium price, whereas in the rational expectations equilibrium framework each agent uses information that he filters out from the equilibrium price, and wants to change his demand after having inferred information of other agents. In the terminology of game theory we could say that his demand is not regret-free.

For the sake of illustration, let us assume that investor A possesses private information that a price of a certain stock will be high in future, and he buys a large number of shares. All other investors possess information that the stock price will be low in future, and after observing the equilibrium price, investor A concludes that his information was wrong, and wants to buy less stocks. Investor A faces a sort of winner’s curse. He has succeeded in buying a large number of stocks only because he has overestimated the true value of the stock. In that sense, the equilibrium price is a price at which none of the agents wants to change the quantity that he demanded (supplied).

$^1$ In this paper we analyse exchange economies in which negative demand is interpreted as supply.
From previous discussion it is not clear how equilibrium price is determined in the rational expectations economy. To clarify this, we will remind ourselves of the process underlying the determination of equilibrium price by the Walrasian auctioneer in a competitive market. In the first step, the Walrasian auctioneer sets a price and then the economic agents tell the auctioneer what quantities how many units they want to supply or demand at that price. If supply is higher than demand the auctioneer lowers the price, and if demand is higher than supply the auctioneer raises the price. After that, the economic agents again tell the Walrasian auctioneer how many units they want to supply or demand. This process continues until supply equals demand.

When agents are asymmetrically informed, the Walrasian auctioneer does not possess private information owned by other agents in the economy. Consequently, an agent will not try to infer private information from the price determined by the auctioneer in the first step (Demange et Laroque, 2001). In the following steps they will have to use very difficult calculations to infer information from the price set by the auctioneer. In order to solve this problem in the rational expectations equilibrium framework, it is assumed that the process of determination of the equilibrium price is not iterative, but rather consists of a single stage. In this case the economic agents inform the auctioneer of their demand and supply functions, i.e. the quantities demanded or supplied at all possible prices. By using the information about demand and supply functions, the auctioneer determines the equilibrium price function and the equilibrium price. Thus, in the rational expectations equilibrium the equilibrium price is determined by the auctioneer.

In rational expectations models it is assumed that the structure of the economy is common knowledge. Each agent knows the endowments of the other agents, their preferences and the distribution functions of random variables. The only information that is private is the information that other agents receive about the

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2 In financial markets agents submit a whole schedule of limit orders, which could be regarded as demand or supply functions.

3 In a subsequent discussion we will explore the difference between price function and equilibrium price in more detail.
realized state of the world. Moreover, in these models it is assumed that all know that all know..., that all agents know the elements of the model⁴.

Let us suppose that the private information set of each agent consists of a signal he receives about the realized state of nature which we denote by \( s^i \). Let us denote the vector of all signals by \( s = (s^1, ..., s^N) \).

**Definition 1.** Rational expectations equilibrium is represented by a price function \( p(s) \to R^+ \), which is a mapping from a set of signals into the price of an asset. Economic agents know the form of \( p(s) \) and they calculate their demand functions by maximizing their expected utility with respect to their signal and information they infer from the equilibrium price \( p = p(s) \). Finally, in equilibrium supply equals demand.

The above definition is too abstract, and we will try to make it a little bit more concrete by explaining the five steps procedure that is used to determine rational expectations equilibrium⁵ (Brunnermeier, 2001).

**First step:** Let there be \( N \) assets in the model. Price function represents mapping from information sets into prices of \( N \) assets: \( p: (I^{\text{inf}}_1, ..., I^{\text{inf}}_N, u) \to R_+^N \). Random variable \( u \) introduces uncertainty into the price function. In the first step, we will, in fact, postulate a whole set of possible price functions, which depend on different undetermined parameters, and the equilibrium values of these parameters will be determined in the last step.

In the **second step** we derive the a posterior beliefs of each agent with respect to the conjectured price function. More precisely, in this step we calculate the conditional expected value and conditional variance for each agent that depend on the values of parameters in the conjectured price function.

In the **third step** we calculate demand functions for all agents that depend on their preferences and their a posterior beliefs.

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⁴ Detailed explanation of the way in which agents discover the structure of the economy is given in Ferrari (2006). Practically, it is assumed that agents update their initial beliefs about the structure of the economy by using the iterative process and they eventually discover the structure of the economy.

⁵ This procedure will be illustrated in the following models.
In the fourth step we equate aggregate demand and aggregate supply and determine the equilibrium price function.

In the fifth step we use the assumption that agents have rational expectations, which means that conjectured price function has to coincide with the equilibrium one. In this step we equate parameters from conjectured and equilibrium price functions.

Before we delve into rational expectations models, we have to explain how agents update their beliefs by using the projection theorem, as well as the form of the demand function when agents have constant absolute risk aversion (CARA preferences) and random variables are normally distributed. We will call this special demand function CARA-Gaussian Demand Function. But, first of all, we need to explain the difference between the fundamental value and the price of an asset.

2. Fundamental value and price of an asset

In this paper we will make a distinction between fundamental value ($v$) and price of a stock ($p$). Fundamental value is determined by the quality of the business activities of a company. Loosely speaking, fundamental value represents the discounted value of all future dividends. The price of a stock is usually different from its fundamental value. If the stock price is higher than the fundamental value a positive bubble exists. On the other hand, if the stock price is lower than the fundamental value a negative bubble exists.

If there is uncertainty about future dividends and investors are risk neutral, fundamental value represents discounted value of expected future dividends. But, if investors are risk averse, they will value cash flows differently depending on their initial endowments. In that case we use martingales to determine fundamental value.

A bubble can occur only under asymmetric information, because investors do not have the same information about future dividends, and they have different

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6 According to Gordon model, fundamental value represents discounted value of infinite series of dividends since capital gains can be ignored in his model.

7 For a further discussion of this problem see Malliaris and Brock (1982), p. 25-28.
estimates of fundamental values. We can say that »objective« fundamental value represents discounted conditional expected value of dividends with respect to all public and private information. Therefore, asymmetrically informed investors are willing to buy or sell assets for a price that is different from fundamental value and a speculative bubble occurs. Moreover, a speculative bubble can exist even if all investors know about its presence, but they believe that other investors do not know it. In that case it is possible that investors are willing to hold the overpriced asset believing that they can sell it at even higher price in the future.

In the following models we will assume that a special player called Nature determines fundamental value before the game begins\(^8\). In other words, fundamental value is exogenously determined\(^9\). Informed investors receive either precise information about fundamental value \(S = v\) or information with some noise \(S = v + \epsilon\). Uninformed investors do not have this information. In this setting a bubble can occur.

For simplicity, in these models it is assumed that in the last period a company pays out a liquidation dividend that is equal to the fundamental value. Therefore, the investor sells a stock if he knows that the price of a stock is higher than the fundamental value and the return he receives is \(p - v\). The investor buys a stock if he knows that the price of the stock is lower than the fundamental value and his return is \(v - p\).

### 3. Bayesian updating, constant absolute risk aversion, CARA-Gaussian demand function and informational efficiency

**Projection theorem**

In the models we will discuss, we will use the assumption that random variables are normally distributed and that agents change their beliefs by using so-called

\(^8\) Harsanyi (1967-68) introduced Nature as a special player in his study of games of imperfect information. The equilibrium concept for these games is called Bayes-Nash equilibrium. In the rational expectations setting equilibrium concept is different, because each agent is a price taker. Needless to say, these models are connected with games of imperfect information because an agent updates his priors in the same way as in games of imperfect information.

\(^9\) By assuming that Nature determines fundamental value, we have made fundamental value exogenous to the model which permits us to ignore flow of dividends.
Bayesian updating. In order to illustrate that process, let $v$ be a $n$-dimensional normally distributed random vector, where $E[v]$ is a vector of expected values. Let $s$ be a $m$-dimensional normally distributed random vector, where $E[s]$ is a vector of expected values. Variance-covariance matrix of random vectors $v$ and $s$ is:

$$
\Sigma = \begin{bmatrix}
E(v - E[v])(v - E[v])' & E(v - E[v])(s - E[s])' \\
E(s - E[s])(v - E[v])' & E(s - E[s])(s - E[s])'
\end{bmatrix} = \begin{bmatrix}
\Sigma_{vv} & \Sigma_{vs} \\
\Sigma_{sv} & \Sigma_{ss}
\end{bmatrix},
$$

where $\Sigma_{vv}$ is a $(n \times n)$ matrix, $\Sigma_{ss}$ is a $(m \times m)$ matrix, $\Sigma_{vs}$ is a $(n \times m)$ matrix and $\Sigma_{sv}$ is a $(m \times n)$ matrix.

Conditional expected value and conditional variance of random vector $v$ with respect to a particular realization of random vector $s$ can be determined by using the projection theorem which is proved in appendix A:

$$
E[v \mid s] = E[v] + \Sigma_{vs} \cdot \Sigma_{ss}^{-1} \cdot (s - E[s]) \quad \text{and} \quad \text{var}[v \mid s] = \Sigma_{vv} - \Sigma_{vs} \cdot \Sigma_{ss}^{-1} \cdot \Sigma_{sv}.
$$

(2)

For the sake of illustration, let $s$ be defined as $s = v + \varepsilon$, where $\varepsilon$ is a $m$-dimensional normally distributed random vector $\varepsilon \sim N(0, \Sigma_{\varepsilon \varepsilon})$, and suppose that $v$ and $\varepsilon$ are independent. By using the previous result we obtain:

$$
E[v \mid s] = E[v] + \Sigma_{vv} (\Sigma_{vv} + \Sigma_{\varepsilon \varepsilon})^{-1} \cdot (s - E[s]) \quad \text{and} \quad \text{var}[v \mid s] = \Sigma_{vv} - \Sigma_{vv}^2 (\Sigma_{vv} + \Sigma_{\varepsilon \varepsilon})^{-1}.
$$

(3)

We will, now, consider a one dimensional case where we have random variables instead of random vectors. Let $v \sim N(E[v], \sigma_v^2)$ and $\varepsilon \sim N(0, \sigma_\varepsilon^2)$ be independent random variables and assume that $s = v + \varepsilon$, then (3) becomes:

$$
E[v \mid S] = E[v] + \frac{\sigma_\varepsilon^2}{\sigma_v^2 + \sigma_\varepsilon^2} \cdot (S - E[S]) \quad \text{and} \quad \text{var}[v \mid S] = \sigma_v^2 - \frac{\sigma_\varepsilon^4}{\sigma_v^2 + \sigma_\varepsilon^2} = \frac{\sigma_v^2 \sigma_\varepsilon^2}{\sigma_v^2 + \sigma_\varepsilon^2}.
$$

(4)

The precision of a random variable is defined as a reciprocal value of its variance. Hence, if $\tau_v = 1/\sigma_v^2$ and $\tau_\varepsilon = 1/\sigma_\varepsilon^2$, then by dividing the numerator and denominator in (4) by $\sigma_v^2 \sigma_\varepsilon^2$, we obtain:

$$
E[v \mid S] = E[v] + \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon} \cdot (S - E[S]) \quad \text{and} \quad \text{var}[v \mid S] = \frac{1}{\tau_v + \tau_\varepsilon}.
$$

(5)
**Constant absolute risk aversion**

The coefficient of absolute risk aversion is defined as a negative ratio of the second and first derivative of utility of money function \( A(W) = -u''(W)/u'(W) \). An agent having the utility function \( u(W) = -e^{-\rho W} \) possesses constant absolute risk aversion \( A(W) = -u''(W)/u'(W) = -\left(-\rho^2 e^{-\rho W}/\rho e^{-\rho W}\right) = \rho \). The coefficient of absolute risk tolerance \( (\eta) \) is defined as a reciprocal value of the coefficient of absolute risk aversion \( \eta = 1/\rho \).

**CARA-Gaussian demand function**

This function is a special type of demand function that is derived by using the assumptions that the investor has constant absolute risk aversion and that his wealth is normally distributed. We will suppose that the investor has the opportunity to invest in two kinds of assets.

The risk-free asset yields a return \( r \), while the fundamental value of the risky asset is \( v \sim N(E[v], \sigma_v^2) \). The investor’s initial endowments of risk-free and risky assets are \( X_F \) and \( X_0 \), respectively. His final demand of the risk-free asset is \( X_F \), and of the risky asset \( X \). Formal derivation of the demand function is in appendix B. CARA-Gaussian demand function has the following form:

\[
X(p) = \frac{E[v] - pr}{\rho \sigma_v^2} = \eta \tau_v [E[v] - pr].
\]  

Demand function (6) has the following properties: (i) quantity demanded is independent of initial endowments; (ii) quantity demanded increases linearly with risk premium \((E[v] - pr)\); (iii) quantity demanded is decreasing in variance \(\sigma_v^2\); (iv) quantity demanded is smaller the higher the risk aversion \(\rho\); (v) when investor becomes risk neutral, i.e. \(\rho \rightarrow 0\), the quantity demanded becomes infinite.

In the following discussion, we will use the assumption that the investor receives a signal \( S = v + \varepsilon \). In this case we will calculate the conditional expected utility and conditional variance of his final wealth. Hence, we will use conditional expected utility \( E[v | S] \) and conditional variance \( \text{var}[v | S] \) instead of \( E[v] \) and \( \sigma_v^2 \) in (6) which can be calculated by using the projection theorem.
**Informational efficiency**

Before proceeding further, we have to define informational efficiency. In his seminal paper Fama (1970) defines three forms of informational efficiency on empirical grounds: weak form efficiency, semi-strong form and strong form. If the market is weak form efficient, the price of an asset reflects only historical information, while if it is semi-strong form efficient, the price of an asset reflects both historical and public information. Finally, if the market is strong form efficient the price of an asset reflects historical, public and private information.

On theoretical grounds we make a distinction between fully revealing and partially revealing equilibrium. In the former the price of an asset reveals all the information in an economy, while in the latter the price of an asset only partially reveals the information in an economy.

After having defined basic concepts used in rational expectations framework, we will examine various models in this setting.

### 4. Fully revealing rational expectations equilibrium

Grossman (1976) considers a model in which equilibrium price reveals all private information. Moreover, equilibrium price is invertible and it reveals sufficient statistic for all information possessed by economic agents. The key assumption which enables that result is that there is no uncertainty on the supply side of the market.

The structure of Grossman’s (1976) model is as follows. There are $I$ investors and investor $i$ is endowed with initial wealth $W_{0i}$. Trade takes place in the present (period 0), and fundamental value, $v$, will be announced in the next period (period 1). Hence, investor's wealth in period 1 is:

$$W_{1i} = (1 + r) \cdot X_{Fi} + v \cdot X_i,$$

where $X_{Fi}$ represents value of risk-free assets at period 0, $X_i$ number of shares bought at period 0, $r$ risk-free rate and $v$ is fundamental value of the stock.$^{10}$

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$^{10}$ Although there are two assets, this is a partial equilibrium analysis, because the price of the risk-free asset is exogenously determined and the price of the risky asset is independent of the price of the risk-free asset.
Budget constraint for \(i\)-th investor is \(W_{0i} = X_{F1} + p \cdot X_i\), where \(p\) is equilibrium price of the stock in period 0. By substituting this result in (7) we obtain:

\[
W_{1i} = (1 + r) \cdot W_{0i} + [v - (1 + r) \cdot p] \cdot X_i.
\]  

(8)

Each of \(I\) investors receives a signal (information) about fundamental value with some noise \(s^i = v + \varepsilon^i\). Random variables \(\varepsilon^i\) are identically, independently, normally distributed \(\varepsilon^i \sim N(0,1)\). Random variable \(v\) is independent of random variable \(\varepsilon^i\) and is also normally distributed \(v \sim N(E[v], \sigma^2_v)\). Each investor possesses constant absolute risk aversion with parameter \(\rho^i\). All investors know the above distribution functions and preferences of other investors and all know that all know... Let us denote by \((j^{\text{inf}})^i\) information set of \(i\)-th investor which consists of his signal \(s^i\) and equilibrium price \(p\). The investor uses his signal and price when calculating his demand. Let us remind ourselves that in rational expectations equilibrium models investors tell the Walrasian auctioneer their demand functions, and then the Walrasian auctioneer calculates the equilibrium price and equilibrium price function. In other words, the equilibrium price is known before the trade begins, although we calculate equilibrium price in the last stage in the procedure for calculating the equilibrium. It is hard to imagine a market that operates in this way, but we can consider the call auction at the opening of NYSE as the approximation for this mechanism, because investors submit limit orders which could be regarded as demand functions, and after that the price that enables the greatest turnover is calculated. Since we know that all random variables are normally distributed and that investors possess constant absolute risk aversion, we can reinterpret the investor’s problem as one of maximizing the following certainty equivalent:

\[
E[W_{1i} \mid (j^{\text{inf}})^i] - 0.5 \cdot \rho^i \cdot \text{var}[W_{1i} \mid (j^{\text{inf}})^i] = (1 + r) \cdot W_{0i} + E[v \mid (j^{\text{inf}})^i] - (1 + r) \cdot p] \cdot X_i - 0.5 \cdot \rho^i \cdot X^2_i \cdot \text{var}[v \mid (j^{\text{inf}})^i]
\]  

(9)

Demand function\(^{11}\) for \(i\)-th investor is:

\[
X_i = \frac{E[v \mid (j^{\text{inf}})^i] - (1 + r) \cdot p}{\rho^i \cdot \text{var}[v \mid (j^{\text{inf}})^i]}.
\]  

(10)

\(^{11}\) The demand function that we use has the property that the quantity demanded can be positive or negative. Negative quantity demanded, in fact, represents supply.
Demand of each investor depends on $s^i$, so equilibrium price depends on the whole vector of signals $s=(s^1,s^2,\ldots,s^l)$, i.e. $p(s)$. Hence, for two different vectors of signals the equilibrium price will be different. Let us denote by $\bar{X}$ the total supply\footnote{As is the case with demand, supply can be positive or negative. If $\bar{X}$ is positive, the investor sells stocks. If it is negative, the investor buys stocks.} of stocks, then in equilibrium the following must hold:

$$\sum_{i=1}^{l} \left\{ \frac{E[v|s^i,P(s)] - (1+r) \cdot P(s)}{\rho^i \cdot \text{var}[v|s^i,P(s)]} \right\} = \bar{X}. \tag{11}$$

The next theorem describes the rational expectations equilibrium. Since this is a competitive model, all investors take the price function as given, i.e. they ignore the impact of their trade on the price. The five step procedure used for deriving equilibrium will be illustrated in the model. We will slightly modify the procedure by combining the last two steps.

**Theorem 1.** Let us assume that price function has the form $p(s) = \alpha_0 + \alpha_S \cdot \bar{S}$, where $\bar{S} = \sum_{i=1}^{l} S^i$. Then

$$\alpha_0 = \frac{\frac{1}{\rho^i} - \sigma_v^2 \cdot \bar{X}}{(1 + I \sigma_v^2) \cdot (1 + r) \sum_{i=1}^{l} \frac{1}{\rho^i}} \quad \text{and} \quad \alpha_S = \frac{I \sigma_v^2}{(1 + I \sigma_v^2) \cdot (1 + r)}, \tag{12}$$

then $p(s)$ is equilibrium price function (solution of equation (11)).

**Proof.** In the first step, let us assume that all investors conjecture that price function has the form $p(s) = \alpha_0 + \alpha_S \cdot \bar{S}$. Since all investors possess rational expectations the conjectured values of the coefficients $\alpha_0$ and $\alpha_S$ coincide with the equilibrium values of those coefficients, permitting the investors to infer $\bar{S}$. Grossman (1976) proves a lemma in which he shows that the individual signal does not contain any additional information that is not contained in the average signal $\bar{S}$, i.e. $E[v|s',\bar{S}]=E[v|\bar{S}]$ and $\text{var}[v|s',\bar{S}]=\text{var}[v|\bar{S}]$. Hence, $\bar{S}$ is a sufficient statistic for $s$. In the second step, we derive aposterior beliefs for each
investor, i.e. conditional expected value and conditional variance of the fundamental value by using the conjecture that \( p(s) = \alpha_0 + \alpha_S \cdot S \):

\[
E[v \mid S] = E[v] + \frac{\sigma_v^2}{\sigma_v^2 + 1/1} \cdot (S - E[v]) = \left( \frac{1}{1 + 1} \right) \cdot E[v] + \frac{1}{1 + 1} \cdot \sigma_v^2 \cdot S, \tag{13}
\]

\[
\text{var}[v \mid S] = \sigma_v^2 - \frac{I \cdot \sigma_v^4}{1 + 1} = \frac{I \cdot \sigma_v^4 + 1 \cdot \sigma_v^4}{1 + 1} = \frac{\sigma_v^2}{1 + 1}. \tag{14}
\]

In the third step, we substitute conjectured price function as well as (13) and (14) into (10) in order to obtain aggregate demand function:

\[
\sum_{i=1}^{I} X_i = \sum_{i=1}^{I} \left\{ \frac{(E[v] + I \cdot \sigma_v^2 \cdot S)}{(1 + I \cdot \sigma_v^2)} - (1 + r) \cdot (\alpha_0 + \alpha_S \cdot S) \right\} \left( \frac{\lambda \cdot \sigma_v^2}{1 + I \cdot \sigma_v^2} \right). \tag{15}
\]

Forth and fifth step are combined. In the fourth step, we equalize aggregate demand and supply. In the fifth step, we assume that investors have rational expectations and that the conjectured price function coincides with the equilibrium price function. This means that coefficients \( \alpha_0 \) and \( \alpha_S \) from the conjectured price function have to be equal to the intercept and slope of the equilibrium price function, respectively. Hence, by using (12) we have that:

\[
\sum_{i=1}^{I} X_i = \sum_{i=1}^{I} \left\{ \frac{(E[v] + I \cdot \sigma_v^2 \cdot S)}{(1 + I \cdot \sigma_v^2)} - (1 + r) \cdot \left( \frac{E[v] \cdot \sum_{i=1}^{I} \frac{1}{\rho_i} - \sigma_v^2 \cdot X}{(1 + I \cdot \sigma_v^2)(1 + r) \cdot \sum_{i=1}^{I} \frac{1}{\rho_i} + I \cdot \sigma_v^2 \cdot \bar{S}} \right) \right\} \tag{16}
\]

The right hand side of (16) is equal to \( \bar{X} \) which proves the theorem.

Having proved that the equilibrium price is invertible, we have seen that it is possible to infer sufficient statistic from the equilibrium price. Furthermore this
means that investors are able to infer all the information contained in the economy. By using this result Grossman (1978) formulated the First Theorem of Welfare Economics under Asymmetric Information. Grossman (1978) shows that in an economy with asymmetrically informed agents there exists a rational expectations equilibrium such that equilibrium allocation cannot be dominated in the sense of Pareto by another allocation chosen by a social planner who possesses all the information in the economy. As Admati (1991) pointed out, fully revealing rational expectations equilibrium is the same as an artificial Walrasian equilibrium in which all agents have the same information before trade begins.

If the equilibrium price is fully revealing, the investor has no incentive to use his own signal, because he can infer sufficient statistic which is superior to his signal. But the following question arises: if none of the investors uses his own signal, how can the equilibrium price be informationally efficient? Furthermore, if information acquisition is not costless, since there exists exogenous cost of information acquisition \( c \), then equilibrium does not exist at all! In that case an investor doesn't have the incentive to buy information, because he can infer sufficient statistic from observing the equilibrium price. On the other hand, if none of the investors buys information, each of them has an incentive to buy information at cost \( c \), because the equilibrium price does not reveal any information. Therefore, if information acquisition is costly, equilibrium does not exist, because each investor has an incentive to deviate.

But if we suppose that supply is represented with a random variable \( u \), instead of being a constant, then the equilibrium price is not fully revealing. High price could be a consequence of an influx of positive news, but could stem from low supply as well. In that case the equilibrium price does not reveal all the information in the economy. This modification is due to Grossman and Stiglitz (1980), who introduced noise traders whose trade is represented by a random variable \( u \).

In the same fashion, Tirole (1982) shows that in a purely speculative market where the gain of one group of rational agents is equal to a loss of another group of rational agents (knowing that they will lose in trade), rational expectations equilibrium does not exist. For this market to operate it is necessary that noise
traders exist. That being the case, rational investors can make a gain in trade at the expense of noise traders.

In the end, it is worth noting that the key assumption that has led to fully revealing equilibrium is that all random variables are normally distributed. De Marzo and Skiadas (1998) show that it is possible to have partially revealing equilibrium, although the supply is constant, if random variables are not normally distributed.

5. Noise traders

As we have shown, the presence of noise traders is essential for financial markets. In the models we study here, it is assumed that these investors trade for reasons that are exogenous to the model. For instance, it could be a desire to achieve a preferred position in liquid assets. In another interpretation these investors trade because they receive some noise which they believe is valuable information (Black, 1986). Whatever the case, the number of shares traded by these investors is independent of the fundamental value of the stock and its price. If these investors were not present, only informed investors would be present. But if one of the investors has more precise information and if that fact is known to the investor having less precise information, he will refuse to take part in an exchange and market failure is inevitable. Hence, the market can operate only in the presence of noise traders. Moreover, in that case it is worthwhile paying a certain amount to obtain information.

Previous analysis leads us to the conclusion that as there is more noise trading markets are more liquid but at the same time less informationally efficient. Empirical research has shown that noise traders trade more with stocks which have lower price. Therefore, stock splits can improve liquidity of the market.

6. Noisy rational expectations equilibrium

As we have stated, Grossman and Stiglitz (1980) modified Grossman’s (1976) model by assuming that supply is a random variable rather than a constant. In

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13 Milgrom and Stokey (1982) formalize this argument in a so-called »No trade theorem«.
14 The model is presented in a simplified version according to Garcia (2004).
other words, noise traders make the supply side of the market. In this case equilibrium price is only partially revealing.

The structure of the model is as follows. There are two groups of rational investors: informed (whose proportion is \( \mu \)) and uninformed (whose proportion is \( 1 - \mu \)). We will treat the trade of noise traders as uncertain supply, and let us denote their trade by a normally distributed random variable \( u \sim N(0, \sigma_u^2) \). If the realized value of \( u \) is positive it means that noise traders sell stocks, and if it is negative noise traders buy stocks. There are two kind of assets: a risky asset (stock) whose fundamental value has a normal distribution\(^\text{15} \) \( v \sim N(0, \sigma_v^2) \), and a riskless asset whose return is normalized to zero. Random variables \( u \) and \( v \) are independent.

All informed investors receive the same signal \( S = v + \varepsilon \), where \( \varepsilon \sim N(0, \sigma_\varepsilon^2) \). Uninformed investors can observe only the price. Both group of investors possess constant absolute risk aversion with parameter \( \rho \). In order to derive equilibrium we will use the five steps procedure.

In the first step we will postulate that investors conjecture a price function of the form \( p = \alpha_0 + \alpha_S \cdot S - \alpha_u \cdot u \), where \( \alpha_0 \), \( \alpha_S \) and \( \alpha_u \) are coefficients\(^\text{16} \). Since all investors have rational expectations, the values of these parameters from the conjectured price function have to coincide with the values of these parameters in the equilibrium price function.

In the second step, we derive aposterior beliefs, i.e. conditional expected values and conditional variances. For the informed investor, the equilibrium price, being only partially revealing, is less informative then his own signal, so informed investors use only their signal in calculating aposterior beliefs. Therefore, conditional expected value and conditional variance for informed investors are:

\[
E[v \mid S, p] = E[v \mid S] = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\varepsilon^2} S \quad \text{and} \quad \text{var}[v \mid S, p] = \text{var}[v \mid S] = \sigma_v^2 - \frac{\sigma_v^4}{\sigma_v^2 + \sigma_\varepsilon^2} = \frac{\sigma_v^2 \cdot \sigma_\varepsilon^2}{\sigma_v^2 + \sigma_\varepsilon^2} .(17)
\]

\(^{15}\) The assumption that \( E[v] = 0 \) is not intuitively appealing, but it is used to simplify the calculations. If we assumed instead that \( E[v] > 0 \), the conclusions would not change, but the calculations would become a little bit more complex.

\(^{16}\) Note that with this price function, the higher the supply, the lower the price.
On the other hand, uninformed investors change their beliefs by using the information they receive from the price. Hence, conditional expected value and conditional variance for uninformed investors are:

\[
E[v \mid p] = \frac{\alpha_S \cdot \sigma_v^2}{\alpha_S^2 \cdot (\sigma_v^2 + \sigma_{\varepsilon}^2) + \alpha_u \cdot \sigma_u^2} \cdot (p - \alpha_0);
\]

\[
\text{var}[v \mid p] = \sigma_v^2 - \frac{\alpha_S^2 \cdot \sigma_v^4}{\alpha_S^2 \cdot (\sigma_v^2 + \sigma_{\varepsilon}^2) + \alpha_u \cdot \sigma_u^2} = \sigma_v^2 \left( \frac{\alpha_S^2 \cdot \sigma_v^2 \cdot \sigma_{\varepsilon}^2 + \alpha_u \cdot \sigma_u^2}{\alpha_S^2 \cdot (\sigma_v^2 + \sigma_{\varepsilon}^2) + \alpha_u \cdot \sigma_u^2} \right). \tag{18}
\]

In the third step, we will derive demand functions. Demand function of informed investors is \( X(S, p) = \frac{E[v \mid S] - p}{\rho \cdot \text{var}[v \mid S]} \), and of uninformed \( Y(p) = \frac{E[v \mid p] - p}{\rho \cdot \text{var}[v \mid p]} \).

In the fourth step, we equalize total demand and total supply \( \mu \cdot X(S, p) + (1 - \mu) \cdot Y(p) = u \):

\[
\mu \cdot \frac{E[v \mid S]}{\rho \cdot \text{var}[v \mid S]} - u = -(1 - \mu) \cdot \frac{E[v \mid p]}{\rho \cdot \text{var}[v \mid p]} + p \left( \frac{\mu}{\rho \cdot \text{var}[v \mid S]} + \frac{1 - \mu}{\rho \cdot \text{var}[v \mid p]} \right). \tag{19}
\]

Both parts on the right hand side of the last equation are linear functions of \( p \), so we can write the right hand side of (19) as \( K \cdot (\alpha_0 + \alpha_S \cdot S - \alpha_u \cdot u) \), where \( K \) is some constant. After rearrangement, the left hand side of (19) becomes \( (\mu / \rho \cdot \sigma_{\varepsilon}^2) \cdot S - u \).

In the fifth step, we match coefficients on the left and right hand sides of (19), that is we use the fact that agents have rational expectations, meaning that the conjectured price function coincides with the equilibrium one. By using this fact we have that \( \alpha_0 = 0 \), \( \alpha_S / \alpha_u = \frac{\mu}{\rho \cdot \sigma_{\varepsilon}^2} \) and \( K \cdot \alpha_u = 1 \). The most important feature of equilibrium is the ratio of coefficients \( \alpha_S / \alpha_u \) which shows the extent to which equilibrium price conveys private information. We will call the inverse of conditional variance conditional precision, and we will denote it as \( \tau[v \mid p] \):

\[
\tau[v \mid p] = \text{var}[v \mid p]^{-1} = \frac{1}{\sigma_v^2} \cdot \left( \frac{\alpha_S^2 \cdot (\sigma_v^2 + \sigma_{\varepsilon}^2) + \alpha_u \cdot \sigma_u^2}{\alpha_S^2 \cdot \sigma_{\varepsilon}^2 + \alpha_u \cdot \sigma_u^2} \right) =
\]
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\[
\frac{1}{\sigma_v^2} \left(1 + \frac{\alpha_S^2 \cdot \sigma_v^2}{\alpha_S^2 \cdot \sigma_E^2 + \alpha_u^2 \cdot \sigma_u^2}\right) = \frac{1}{\sigma_v^2} + \frac{1}{\sigma_v^2 + (\alpha_u / \alpha_S)^2 \cdot \sigma_u^2}. \tag{20}
\]

It is obvious that the higher the conditional precision, the lower the conditional variance. Conditional precision is a measure of informational efficiency and it is increasing in \(\alpha_S / \alpha_u = \mu / (\rho \cdot \sigma_E^2)\). Therefore, we can conclude that: (1) by increasing the proportion of informed investors (\(\mu\)) we increase informational efficiency; (2) lower quality of information (higher \(\sigma_E^2\)) decreases informational efficiency; (3) higher risk aversion of informed investors causes lower informational efficiency because informed investors trade less aggressively; (4) lower variability of supply (lower \(\sigma_S^2\)) increases informational efficiency.

In further analysis of the model, Grossman and Stiglitz (1980) assume that the number of informed investors is endogenously determined. In other words, the investor will pay an amount \(c\) for a signal and he will become informed if his \textit{ex ante} expected utility\(^{17}\) when he pays that amount and receives a signal is higher then when he stays uninformed. We have seen that as the proportion of informed investors increases, informational efficiency increases as well, which enables uninformed investors to infer more information from the price. If the equilibrium price revealed all the private information then none of the investors would be willing to pay an amount \(c\) to buy a signal. Each investor would behave as a free rider, assuming that other investors will buy a signal while he will infer information freely from the equilibrium price. But since all investors reason in the same way, none of them would buy a signal. In this case it is not possible that the equilibrium price is fully revealing, and that each investor has an incentive to buy a signal. In other words, equilibrium for this game does not exist. This result is known as Grossman-Stiglitz’s paradox.

\(^{17}\) It is important to make a distinction between \textit{ex ante} and \textit{ex post} expected utility. \textit{Ex post} expected utility represents expected utility after the informed agent receives a signal about the realized state of the world, while \textit{ex ante} expected utility represents expected utility of the investor who will be informed before he receives a signal. More formally, ex ante expected utility represents definite integral of \textit{ex post} expected utility with respect to all states of the world.
7. Information aggregation

In contrast to Grossman-Stiglitz's (1980) model where all investors receive the same signal, in Hellwig's (1980) model each investor receives a different realization of the signal $S^i = v + \epsilon^i$, where $\epsilon^i \sim N(0,(\sigma_{\epsilon}^2))$. In other words, while in Grossman-Stiglitz's (1980) model all signals are perfectly positively correlated, in Hellwig's (1980) model signals are positively, but not perfectly positively correlated, because each signal comprises one common and one idiosyncratic component. In this model the equilibrium price has one additional role: it has to aggregate different information in the economy.\(^{18}\)

The structure of the model is as follows. There are \(I\) investors each having constant absolute risk aversion with parameter $\rho^i$. We will denote the coefficient of absolute risk tolerance by $\rho^i \eta^i = 1/\rho^i$. Total supply is denoted by a random variable $u$. We will use the following notation: $\Delta S^i = S^i - E[S^i]$, $\Delta u = u - E[u]$, $\Delta p = p - E[p]$. In the first step, let us postulate that investors conjecture that price function has the following form:

$$ p = \alpha_0 + \sum_i \alpha^i S^i + \alpha_u \Delta u. \quad (21) $$

The information set of each investor comprises his signal and his equilibrium price. In calculating his a posterior beliefs, the investor uses both equilibrium price and his signal, because equilibrium price aggregates information possessed by other investors which is different from information possessed by a particular investor. In the second step, we derive a posterior beliefs for each investor

$$ E[v | S^i, p] = E[v] + \beta^i_S \Delta S^i + \beta^i_p \Delta p \quad \text{and} \quad \text{var}[v | S^i, p] = 1/\tau^i [v | S^i, p], $$

where $\beta^i_S$ and $\beta^i_p$ are regression coefficients. Third, the aggregate demand of each investor is $X_i(p) = \eta^i \tau^i [v | S^i, p] \{E[v | S^i, p] - p(1+r)\}$. The return to risk-free asset is normalized to $r = 0$. Fourth, from the condition for equilibrium we obtain that\(^{19}\):

$$ \sum_i \eta^i \tau^i [v | S^i, p] \{E[v] + \beta^i_S \Delta S^i - \alpha_0 \beta^i_p + [\beta^i_p - 1]p\} = u. \quad (22) $$

\(^{18}\) This interpretation of the model is adapted from Brunnermeier (2003).

\(^{19}\) We use the fact that $E[p] = \alpha_0$. 

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constant absolute risk aversion with parameter \( \rho \). The structure of the model is as follows. There are random variable realizations of the signal. We use the fact that this interpretation of the model is adapted from Brunnermeier (2003).

In contrast to Grossman-Stiglitz’s (1980) model where all investors receive the same signal, in Hellwig’s (1980) model each investor receives a different signal. In Grossman-Stiglitz’s (1980) model signals are perfectly positively correlated, whereas in Hellwig’s (1980) model signals share the same distribution function normalized to the coefficient of risk aversion \( 1/\eta \), and idiosyncratic components of their signals share the same distribution function \( \epsilon^j \sim N(0, \sigma^2_\epsilon) \). Let us assume that investors conjecture that price function has the form:

\[
\Delta p = \alpha_S \sum_{i} \frac{1}{\beta_p} \Delta S^i + \alpha_u \Delta u.
\]  

In order to simplify the model, we will assume that all investors have the same coefficient of risk aversion \( \rho = 1/\eta \), and idiosyncratic components of their signals share the same distribution function \( \epsilon^j \sim N(0, \sigma^2_\epsilon) \). Let us assume that investors conjecture that price function has the form:

\[
\alpha_0 = \frac{\sum (\eta^j \tau^j[v | S^i, p]) [E[v] - \alpha_0 \beta_p^j + \beta^j_S \Delta S^i] - E[u] - \Delta u}{\sum_i (1 - \beta_p^j) \eta^j \tau^j[v | S^i, p]}, \quad \alpha_S^j = \frac{\eta^j \tau^j[v | S^i, p] \beta^j_S}{\sum_i (1 - \beta_p^j) \eta^j \tau^j[v | S^i, p]}, \quad \alpha_u^j = \frac{-1}{\sum_i (1 - \beta_p^j) \eta^j \tau^j[v | S^i, p]}.
\]

In order to determine coefficients \( \beta_S \) and \( \beta_p \), we will use the projection theorem in matrix form:

\[
E[v | S^i, p] = E[v] + [\text{cov}(v, S^i) \text{ cov}(v, p)] \cdot \Sigma^{-1}(S^i, p) \cdot \begin{bmatrix} \Delta S^i \\ \Delta p \end{bmatrix},
\]

where \( \Sigma^{-1}(S^i, p) \) is the inverse of the variance-covariance matrix between signals and price, and \( \text{cov}(\cdot, \cdot) \) is covariance between random variables. Variance-covariance matrix and its inverse matrix are:
\[ \Sigma(s^i, p) = \begin{bmatrix} \sigma_v^2 + \sigma_e^2 & \alpha_s (\sigma_v^2 + \frac{1}{I} \sigma_e^2) \\ \alpha_s (\sigma_v^2 + \frac{1}{I} \sigma_e^2) & \alpha_s^2 (\sigma_v^2 + \frac{1}{I} \sigma_e^2) + \alpha_u^2 \sigma_u^2 \end{bmatrix}, \]

\[ \Sigma^{-1}(s^i, p) = \frac{1}{D} \begin{bmatrix} \alpha_s^2 (\sigma_v^2 + \frac{1}{I} \sigma_e^2) + \alpha_u^2 \sigma_u^2 & -\alpha_s (\sigma_v^2 + \frac{1}{I} \sigma_e^2) \\ -\alpha_s (\sigma_v^2 + \frac{1}{I} \sigma_e^2) & \sigma_v^2 + \sigma_e^2 \end{bmatrix}, \quad (28) \]

\[ D = (\sigma_v^2 + \sigma_e^2) \left[ (\sigma_v^2 + \frac{1}{I} \sigma_e^2) + \alpha_u^2 \sigma_u^2 \right] - \alpha_s^2 \left[ \sigma_v^2 + \frac{1}{I} \sigma_e^2 \right]^2 \]

is determinant. We have that \( \text{cov}(v, s^i) = \sigma_v^2 \) and \( \text{cov}(v, p) = \alpha_s \sigma_v^2 \). By substituting these results in (27) we obtain:

\[ E[v | s^i, p] = E[v] + \frac{1}{D} \sigma_v^2 \alpha_s \sigma_v^2 \left[ \alpha_s^2 (\sigma_v^2 + \frac{1}{I} \sigma_e^2) + \alpha_u^2 \sigma_u^2 - \alpha_s (\sigma_v^2 + \frac{1}{I} \sigma_e^2) \right] \Delta s^i, \quad (29) \]

\[ E[v | s^i, p] = E[v] + \frac{1}{D} \alpha_s^2 \sigma_u^2 \sigma_v^2 \Delta s^i + \frac{1}{D} \alpha_s I \frac{1}{I} \sigma_v^2 \sigma_e^2 \Delta p. \quad (30) \]

From (30) we conclude that \( \beta_s = \frac{1}{D} \alpha_s^2 \sigma_u^2 \sigma_v^2 \) and \( \beta_p = \frac{1}{D} \alpha_s I \frac{1}{I} \sigma_v^2 \sigma_e^2 \).

Conditional variance is:

\[ \text{var}[v | s^i, p] = \sigma_v^2 - \left[ \text{cov}(v, s^i) \cdot \text{cov}(v, p) \right] \Sigma^{-1}(s^i, p) \left[ \begin{bmatrix} \text{cov}(v, s^i) \\ \text{cov}(v, p) \end{bmatrix} \right] = \frac{1}{D} \left[ D \sigma_v^2 + \alpha_s^2 \sigma_u^2 \sigma_v^2 - \alpha_s \sigma_v^2 \frac{1}{I} \sigma_e^2 \sigma_v^2 \right], \quad (31) \]

\[ \text{var}[v | s^i, p] = \frac{1}{D} \left[ D \sigma_v^2 + \alpha_s^2 \sigma_u^2 + \frac{1}{I} \sigma_v^2 \sigma_e^2 \right] \sigma_v^2 \sigma_e^2 \]. \quad (32) \]

By substituting \( \beta_s \) and \( \beta_p \) in (26) we obtain:

\[ \alpha_s = \frac{\alpha_s^2 \sigma_u^2 \sigma_v^2}{D - \alpha_s \frac{I}{I} \sigma_v^2 \sigma_e^2}, \quad \alpha_u = -\rho \frac{\alpha_s^2 \frac{I}{I} \sigma_e^2 + \alpha_u^2 \sigma_u^2 \sigma_e^2}{\frac{D}{I} \left( D - \alpha_s \frac{I}{I} \sigma_v^2 \sigma_e^2 \right)} \]. \quad (33)
We seek the solution in the form of \( h = -\frac{\alpha_u}{\alpha_S} \), because price function could be written as \( \frac{p}{\alpha_S} = \Sigma_i \frac{1}{I} \Delta S_i + \frac{\alpha_u}{\alpha_S} u_i \):

\[
h = \frac{\alpha_u}{\alpha_S} = \frac{\rho \left( \alpha_u^2 \sigma_u^2 + \alpha_S^2 \frac{I-1}{I^2} \sigma_e^2 \right) \sigma_v^2 \sigma_u^2}{I \alpha_u^2 \sigma_v^2 \sigma_u^2 \sigma_e^2} = \frac{\rho \left( h^2 \sigma_u^2 + \frac{I-1}{I^2} \sigma_e^2 \right) \sigma_v^2 \sigma_e^2}{I h^2 \sigma_e^2 \sigma_u^2} ,
\]

\[
h = \frac{1}{I} \left( \rho \frac{\sigma_v^2}{h^2} + \frac{\rho}{I} \frac{I-1}{I} \frac{\sigma_e^4}{\sigma_u^2} \right). \tag{35}
\]

The left hand side of (35) is increasing in \( h \), while the right hand side is decreasing in \( h \), which leads us to the conclusion that there exists a unique \( h \) that solves (35). We can also see that \( h \) is increasing in \( \rho \).

Let us concentrate on parameter \( \alpha_S \). Rearranging (33) we obtain that

\[
\alpha_S = \frac{1}{D} \left[ \alpha_u^2 \sigma_u^2 \sigma_v^2 + \alpha_S^2 \frac{I-1}{I} \sigma_e^2 \sigma_v^2 \right].
\]

By substituting out \( D \) and by multiplying both numerator and denominator by \( \alpha_S^2 \) we obtain:

\[
\alpha_S = \frac{\left( \frac{\alpha_u^2}{\alpha_S^2} \right) \sigma_v^2 \sigma_u^2 + \frac{I-1}{I} \sigma_v^2 \sigma_e^2}{\frac{I-1}{I} \left( \sigma_v^2 + \frac{1}{I} \sigma_e^2 \right) \sigma_e^2 + \left( \frac{\alpha_u^2}{\alpha_S^2} \right) \sigma_u^2 \left( \sigma_v^2 + \sigma_e^2 \right)} = \frac{h^2 \sigma_v^2 \sigma_u^2 + \frac{I-1}{I} \sigma_v^2 \sigma_e^2}{\frac{I-1}{I} \left( \sigma_v^2 + \frac{1}{I} \sigma_e^2 \right) \sigma_e^2 + h^2 \sigma_u^2 \left( \sigma_v^2 + \sigma_e^2 \right)} . \tag{36}
\]

Since we have proved the existence of unique \( h \) that solves (35), we can conclude that there exists unique \( \alpha_S \). In that way we have proved the uniqueness of equilibrium, even though we are not able to derive closed form expressions for \( \alpha_S \) and \( \alpha_u \).

We will next derive a measure of informational efficiency. We know that

\[
\text{var}[\nu | S', p] = \frac{1}{D} \left[ \alpha_S^2 \frac{I-1}{I^2} \sigma_e^2 + \alpha_u^2 \sigma_u^2 \right] \sigma_v^2 \sigma_u^2 \text{ and } \alpha_S = \frac{1}{D} \left[ \alpha_S^2 \frac{I-1}{I} \sigma_v^2 \sigma_e^2 + \alpha_u^2 \sigma_u^2 \sigma_v^2 \right].
\]

It follows that:
\[
\alpha_S = \text{var}[v \mid S^i, p] \cdot \frac{1}{I^2} \begin{bmatrix}
\alpha^2_S & I - 1 & \alpha^2_S \\
I^2 & I - 1 & I^2
\end{bmatrix} \cdot \frac{I^2}{I - 1} \cdot \frac{1}{\alpha^2_S} = \text{var}[v \mid S^i, p] \cdot \frac{1}{\alpha^2_S} \begin{bmatrix}
\frac{I^2}{I - 1} h^2 \sigma^2_u + I \sigma^2_v \\
\frac{I^2}{I - 1} h^2 \sigma^2_u + I \sigma^2_v
\end{bmatrix}, \tag{37}
\]

\[
\alpha_S = \text{var}[v \mid S^i, p] \cdot \frac{1}{\sigma^2_v} \begin{bmatrix}
\frac{I^2}{I - 1} h^2 \sigma^2_u + \sigma^2_v + (I - 1) \sigma^2_v \\
\frac{I^2}{I - 1} h^2 \sigma^2_u + \sigma^2_v + (I - 1) \sigma^2_v
\end{bmatrix} = \text{var}[v \mid S^i, p] \cdot \frac{1}{\sigma^2_v} \begin{bmatrix}
1 + (I - 1) \sigma^2_v \\
1 + (I - 1) \sigma^2_v
\end{bmatrix}. \tag{38}
\]

By dividing the expressions in parentheses in the numerator and denominator of the last equation by \( \sigma^2_v \sigma^2_u \), we finally obtain \(^20\):

\[
\alpha_S = \text{var}[v \mid S^i, p] \cdot \frac{\tau_u}{\tau_u + \frac{I^2}{I - 1} h^2 \tau_v} = \text{var}[v \mid S^i, p] \cdot \tau_v [1 + (I - 1) \Phi], \tag{39}
\]

where \( \Phi = \frac{\tau_u}{\tau_u + \frac{I^2}{I - 1} h^2 \tau_v} \). From (39) we see that \( \frac{1}{\text{var}[v \mid S^i, p]} = \tau_v [1 + (I - 1) \Phi] / \alpha_S \).

From this result we can deduce that \( \Phi \) represents a measure of informational efficiency. Hellwig (1980) also proved that \( \text{var}[v \mid S^i, p]^{-1} = \tau_v + \tau_v + (I - 1) \Phi \tau_v \). From that result it is obvious that \( \Phi \) measures the extent to which equilibrium price aggregates private information.

Several aspects of comparative statics are worth mentioning. When variability of supply tends to infinity (high noise limit), equilibrium price does not aggregate information at all: \( \sigma^2_u \to \infty \) \( (\tau_u \to 0) \Rightarrow \Phi \to 0 \). In other words, prior beliefs coincide with aposterior beliefs \( \text{var}[v \mid S^i, p]^{-1} = \tau_v + \tau_v + (I - 1) \Phi \tau_v \). On the other hand, when variability of supply tends to zero (low noise limit), equilibrium price completely aggregates private information: \( \sigma^2_u \to 0 \) \( (\tau_u \to \infty) \Rightarrow \Phi \to 1 \). In that case \( \text{var}[v \mid S^i, p]^{-1} = \tau_v + I \tau_v \). We can also deduce that the higher the coefficient of risk aversion, the lower the informational efficiency, because \( h(\cdot) \) is increasing function of \( \rho \), and therefore \( \Phi(\cdot) \) is decreasing function of \( h \). In the limiting

\(^{20}\) As usual \( \tau \) represents precision of a random variable.
case when $\rho \to 0 \ (h \to 0) \Rightarrow \Phi \to 1$. Therefore, when investors are almost completely risk neutral they trade more aggressively, offsetting the impact of noise traders. Finally, it is possible to show that informational efficiency is increasing in signal precision $\tau_{\varepsilon}$.

Admati (1985) considers information aggregation in a multi-asset model, with $N$ securities. In a very complicated paper, Admati (1985) uses partitioned matrices\(^{21}\) and derives rational expectations equilibrium. The main finding of the paper is that if noise traders’ trades are correlated across markets, the uncertainty in the supply of only one asset can prevent the prices of other assets, whose supply is fixed, of being fully revealing. This is an explanation of Grossman-Stiglitz’s paradox in the multi-asset model.

8. Endogenous signal precision

In Hellwig’s (1980) model each agent receives a signal of exogenous quality. Verrechia (1982) assumes that the signal quality is endogenously determined, meaning that each agent decides about the quality of his signal. Each investor can buy a signal $s^i = v + \varepsilon^i$, where $\varepsilon^i \sim N(0, 1/\tau_{\varepsilon})$. The cost of buying a signal is increasing, convex function $(c'(\tau_{\varepsilon}) > 0, c''(\tau_{\varepsilon}) \geq 0))$. In his model all investors possess constant absolute risk tolerance. The coefficient of constant absolute risk tolerance belongs to interval $[\bar{\eta}, \bar{\eta}]$ with the probability density function $g(\eta)$.

Verrechia (1982) shows that the quality of the signal is increasing function of agent’s risk tolerance. This is very intuitive, since we know from the CARA-Gaussian demand function that the quantity demanded is increasing function of agent’s risk tolerance. Therefore, the more risk tolerant agent buys a signal of a higher quality since his potential gain (loss) is higher than that of an agent with lower risk tolerance.

By using comparative statics Verrechia (1982) shows that the decrease in the cost of information $c'(\tau_{\varepsilon}) \leq c'(\tau_{\varepsilon})$, increases informational efficiency. The second result is that if the distribution function $g(\eta)$ is replaced with the new

\(^{21}\) The interested reader should read chapter 1.2. from Sydsæter, Hammond, Seierstad and Strøm, (2005) that deals with partitioned matrices.
one, so that first order stochastically dominates the old one, informational efficiency will be higher, because agents will buy more precise signals.

9. Stock market crashes in rational expectations equilibrium models

A stock market crash takes place in the case of a substantial decline of stock prices. The speed of recovery of the market after a crash depends on the factors that caused the crash. There are five group of models that explain market crashes.

According to proponents of the first group of models, the decline in prices can be due to temporary reduction of liquidity. For instance, some investors can mistakenly interpret huge number of sells driven by dynamic trading strategy as sells driven by bad news. Dynamic trading strategy is the automatic sale of stocks when a stock price falls below some prespecified level, or automatic purchase of stocks when the stock price rises above some prespecified level. Because of this misinterpretation of dynamic trading, investors are less willing to buy shares and the market dries up. In this case price decline is temporary in nature and fast recovery of the market can be expected.

The second class of models attributes crashes to lumpy information aggregation. In this class of models less informed investors suddenly reveal information possessed by better informed investors by observing previous price changes. When they discover that information, less informed investors drastically change their demand, and a market crash occurs. The price decline is usually preceded by a steady increase in prices, and price decline represents a correction. Therefore, the market is not expected to bounce back quickly.

The third class of models argues that stock market crashes could be caused by the existence of multiple equilibria. In this case price function is, in fact, price correspondence because several equilibrium prices exist for the same values of

\[ \text{Formally, for all } \eta \in [\underline{\eta}, \overline{\eta}], \text{ new probability density function } \bar{g}(\eta) \text{ first order stochastically dominates probability density function } g(\eta) \text{ if the following holds } \int_{\underline{\eta}}^{\overline{\eta}} \bar{g}(\eta) d\eta \leq \int_{\underline{\eta}}^{\overline{\eta}} g(\eta) d\eta. \text{ For the formal proof that first order stochastic dominance implies preference and vice versa, see Hadar and Russell (1969).} \]
parameters. The public announcement of some information that does not even have to be related to the fundamental value of the asset (sunspot), could affect the beliefs of agents in such a way that the economy shifts from high asset price equilibrium to another equilibrium with a low price.

The fourth class of models shows that market crashes are due to the existence of a positive speculative bubble. Let us remind ourselves that a positive bubble occurs when the stock price is higher than its fundamental value. The existence of a positive bubble is mutually known by all traders, but a particular trader does not know that other traders are aware that a bubble exists. In other words, the existence of a bubble is not common knowledge. Therefore, an investor believes that he can sell the stock to someone else at a higher price and that explains why he is willing to buy overpriced stock. At a certain point in time the bubble bursts, and a huge price decline occurs. In this case fast recovery is not expected.

The fifth class of models attributes a market crash to informational cascade. In informational cascade the investor puts more weight on the information that he infers from trading history and neglects his own signal. If the informational cascade is shattered by an informational event that will persuade investors that the asset is overpriced, investors will intensively sell their shares and a sharp price decline is inevitable.

**9.1. Temporary reduction of liquidity**

In Grossman's (1988) model risk aversion of some of the investors increases as their wealth declines. Therefore, these investors use dynamic trading strategy, i.e. they automatically sell stocks when a price drops below a certain threshold. On the other side of the market there are some speculators willing to buy stocks at a low price and wait for a price increase in later periods. These speculators use part of their wealth to buy shares, and invest other part of their wealth in other projects. If speculators underestimate the amount of dynamic sales, they do not have enough funds to buy all the stocks that are subject to dynamic sale. This can lead to liquidity shortage and a decline in prices. Recovery can be expected when speculators free up money that they invested in other projects, and price decline is temporary in nature.
9.2. Lumpy information aggregation

The stock market crash of 1987 has been the subject of much research. The peculiarity of that crash was that it was not preceded by a major influx of bad news. Based on the absence of extremely bad news, one would expect to see only a slight decline in prices. Romer’s (1993) model attributes the 1987 crash to sudden information revelation.

Romer (1993) considers a two period model in which trade takes place in the first period and in which the equilibrium price in the first period does not aggregate all private information. In his model a small commonly known supply shift can reveal information that is not aggregated in the first period equilibrium price to a part of the investors. In other words, a decline of the equilibrium price is possible, despite the absence of a new influx of bad information.

As usual, we will suppose that fundamental value is normally distributed $v \sim N(E[v], \sigma_v^2)$. The supply is random and denoted by a random variable $u \sim N(0, \sigma_u^2)$. There is another risk-free asset that yields zero return.

Each investor has constant absolute risk aversion with parameter $\rho$. Final wealth of $i$-th investor is $W_i = X_i v + X_{Fi}$, where $X_i$ and $X_{Fi}$ are final demands for risky and risk-free assets respectively. Each investor can receive one of three signals $S_i = v + \varepsilon_i$ ($i = 1, 2, 3$), where $\varepsilon_i$ has normal distribution $\varepsilon_i \sim N(0, (\sigma_{\varepsilon_i}^2))$. Random variable $\varepsilon_2$ is defined as $\varepsilon_2 = \varepsilon_1 + \theta_2$, and random variable $\varepsilon_3$ is defined as $\varepsilon_3 = \varepsilon_2 + \theta_3$. Hence, $S_1$ is the most precise signal, and $S_3$ is the least precise signal.

There are two equally likely states of the world of signal distribution. In the first state half of the investors receive signal $S_1$, and half receive signal $S_2$. In the second state half of the investors receive signal $S_2$, and half receive signal $S_3$. Investors receiving a signal $S_i$ observe the same realisation of the signal. Hence, only traders who have received a signal $S_2$ do not know whether the other half of traders have received the more precise signal $S_1$ or the less precise signal $S_3$. Investors who have received signal $S_1$ know that they possess the most precise signal and do not infer additional information from the equilibrium price in the first period. Since the first period equilibrium price does not aggregate all
information, after the first period investors with signal $S^2$ still do not know whether the other half of the investors have signal $S^1$ or $S^3$. Let us suppose that after the trade is completed in the first stage but before the trade begins in the second stage a commonly known supply shift occurs. The change of equilibrium price enables investors with signal $S^2$ to infer the slope of the demand function of the other investors.$^{23}$ Namely, investors with signal $S^1$ have demand function $[E(v | S^1) - p] / \rho \cdot \text{var}(v | S^1)$, where:

$$
E[v | S^1] = E[v] + [\sigma_v^2 / (\sigma_v^2 + \sigma_{\epsilon_1}^2)] \cdot (S^1 - E[v]) \text{ and } \text{var}[v | S^1] = \sigma_v^2 - \frac{\sigma_v^2 \cdot \sigma_{\epsilon_1}^2}{\sigma_v^2 + \sigma_{\epsilon_1}^2}.
$$

Therefore, if the absolute value of the slope of the demand function of the other investors is $1 / (\rho \cdot \text{var}[v | S^1])$, then investors with signal $S^2$ know that all other investors have signal $S^1$. On the other hand, investors with signal $S^3$ face higher uncertainty with regard to fundamental value, so the absolute value of the slope of their demand function is lower. When investors with signal $S^2$ reveal whether the other half of investors have the more or less precise signal, their demand changes dramatically. By using the simulation analysis, Romer (1993) shows that the demand function of investors with signal $S^2$ is considerably different depending on the quality of the signal the other investors have. This change of demand leads to a considerable change of stock price in the second period and a price discontinuity occurs. In other words, the trading process itself can lead to a price decline, despite the absence of new bad information.

9.3. Multiple equilibria

Gennotte and Leland (1990) have a different explanation of the 1987 market meltdown. The main assumption in their paper is a distinction between expected and unexpected supply change. The main conclusion of the paper is that if there is uncertainty about the exact amount of dynamic trading, then multiple equilibria may arise. If the economy shifts from one equilibrium to another, a substantial price decline can occur.

$^{23}$ The assumption that supply shift is common knowledge is essential. If supply shift was random, investors with signal $S^2$ could not infer the slope of the demand function of other investors.
There are three types of investors in the model. Uninformed investors only observe price, $p$. Value informed traders receive a signal about fundamental value $S_i = v + e_i$. Supply informed investors receive a signal about part of future supply $ue$ and observe the price.

Aggregate supply in the market is $\bar{u} + uu + ue + \Delta_d(p) \equiv u + \Delta_d(p)$, where $\bar{u}$ is common knowledge, $ue \sim N(0, \sigma_{ue}^2)$ is a part of supply that is known to supply informed investors, $uu \sim N(0, \sigma_{uu}^2)$ is part of supply that is not known to anybody, and $\Delta_d(p)$ is supply that stems from investors who use dynamic trading.

Gennotte and Leland (1990) proved that in the absence of dynamic trading price function is continuous and well defined, which rules out discontinuous price changes (market crashes). Moreover, they showed that even when dynamic trading is present but $\Delta_d(p)$ is linear function, the price function is continuous and crashes are not possible. Only when $\Delta_d(p)$ is a nonlinear function the price function is discontinuous and multiple equilibria arise. In that case a sharp price decline is possible if the economy shifts from one equilibrium to another.

Gennotte and Leland (1990) use simulation analysis and derive three excess demand curves. The curve labeled $XDA$ is derived under assumption that the exact amount of dynamic trading is common knowledge, the curve labelled $XDP$ is relevant only when supply informed traders know the exact amount of dynamic trading, and the curve $XD_U$ is relevant when all investors are ignorant of the exact amount of dynamic trading (figure 1).
There are three types of investors in the model. Uninformed investors only observe price $p$. Value informed traders receive a signal about fundamental value $\epsilon$. Supply informed investors receive a signal about part of future supply $\mu$ and observe the price. Aggregate supply in the market is $p = u_1 + \Delta + \Delta + \Delta + \Delta$, where $u$ is common knowledge, $u$ is part of supply that is known to supply informed investors, $u$ is part of supply that is not known to anybody, and $\Delta$ is supply that stems from investors who use dynamic trading.

Gennotte and Leland (1990) proved that in the absence of dynamic trading price function is continuous and well defined, which rules out discontinuous price changes (market crashes). Moreover, they showed that even when dynamic trading is present but $\Delta$ is linear function, the price function is continuous and crashes are not possible. Only when $\Delta$ is a nonlinear function the price function is discontinuous and multiple equilibria arise. In that case a sharp price decline is possible if the economy shifts from one equilibrium to another.

Gennotte and Leland (1990) use simulation analysis and derive three excess demand curves. The curve labeled $\mathcal{XDA}$ is derived under assumption that the exact amount of dynamic trading is common knowledge, the curve labelled $\mathcal{XDP}$ is relevant only when supply informed traders know the exact amount of dynamic trading, and the curve $\mathcal{XDU}$ is relevant when all investors are ignorant of the exact amount of dynamic trading (figure 1).

Figure 1. *Multiple equilibria and market breakdown*

Zero excess demand (market equilibrium) is represented by a vertical curve. From the figure we see that multiple equilibria are possible only when all investors don’t know the exact amount of dynamic trading. The inflow of bad information, which does not have to be of high magnitude, could move all three curves to the left causing a shift to another equilibrium with a substantially lower price. Equilibrium is locally stable if the excess demand curve has a negative slope at the point of intersection with the vertical line. In that case, if the price is slightly higher or lower than the equilibrium one, market forces will govern the price to locally stable equilibrium level. Otherwise, if the excess demand curve has a positive slope at the point of intersection with the vertical line, the equilibrium is locally unstable. In that case if the price is slightly higher or lower than the equilibrium one, market forces will govern the price away from the locally unstable equilibrium level. The conclusion of Gennotte and Leland (1990) is that in order to prevent a stock market crash, regulators should publicly announce information about dynamic trading.

9.4. Informational avalanche

Lee (1998) argues that informational cascade can trigger a market breakdown. In informational cascade investors put greater emphasis on previous trading history than on their own signal. For instance, if previously a lot of investors
bought stocks and a particular investor has a bad signal, he will ignore his signal and follow the decisions of previous investors. But if an investor with an extremely negative signal concludes that the asset is overpriced he will sell the stocks. When other investors reach the same conclusion, this will trigger an avalanche of sell orders and the price will fall sharply. Thus a market crash represents a correction of agents' beliefs which were inconsistent with their private information. The price after the crash is likely to be close to the fundamental value of asset.

10. Concluding remarks

The models that we have discussed are based on several assumptions. In this section we will examine truthfulness of these assumptions. The first assumption is that agents maximize expected utility. The theory of expected utility has been criticized by behavioural economists. By using experimental methodology they have shown that the assumptions of the theory of expected utility are not always satisfied. Nevertheless, the theory of expected utility is widely used in microeconomics, because the use of the alternative concept for the characterization of preferences under uncertainty would make it impossible to derive of the closed form solution.

The second assumption is that random variables are normally distributed. We know from empirical facts that asset returns are asymmetrically distributed. Despite that fact, the normality assumption is used to simplify derivation of a posterior beliefs by using the projection theorem.

The third assumption is that all agents have the same coefficient of constant absolute risk aversion (except in one model). This assumption is used to eliminate income effect and to simplify the demand function. We argue that classification of investors in different groups according to their risk aversion would make models more realistic.

The fourth assumption is that agents have rational expectations, meaning that they know the structure of the exchange economy and that they are capable of making sometimes very complex calculations, which is not a very realistic assumption.
The fifth assumption is that informed agents do not affect the equilibrium price with their trade. That assumption is in sharp contrast not only with empirical facts from financial markets but also with modeling procedures. As Hellwig (1980) noticed, investors behave schizophrenically. On the one hand, they do not take into account the impact of their trade on the equilibrium price, but on the other hand they try to infer the information of other agents from the equilibrium price, which means that they implicitly believe that the trades of other investors influence the equilibrium price. By relaxing the price taking assumption we obtain another class of models of asymmetric information in financial markets called strategic share auctions\textsuperscript{24}. Models in this class are extremely complicated which makes one ask oneself: is it worthwhile relaxing the price taking assumption?

Despite the problematic assumptions, this concept has made a significant progress from the Walrasian equilibrium concept because rational expectations equilibrium sheds a light on the process of learning in competitive markets, which had to be ignored by Léon Walras. But in order to make rational expectations equilibrium concept tractable, quite a few simplifying assumptions are needed.

**Appendix**

**Proof of the projection theorem**\textsuperscript{25}

Let $\mathbf{v}$ be a $(n \times 1)$ normally distributed random vector and $\mathbf{s}$ normally distributed $(m \times 1)$ random vector which is independent of $\mathbf{v}$. The two vectors have the property that $E[\mathbf{v}] = E[\mathbf{s}] = \mathbf{0}$. The variance-covariance matrix for these two vectors is:

$$
\Sigma = \begin{bmatrix}
E(\mathbf{v}\mathbf{v}') & E(\mathbf{v}\mathbf{s}') \\
E(\mathbf{s}\mathbf{v}') & E(\mathbf{s}\mathbf{s}')
\end{bmatrix} = \begin{bmatrix}
\Sigma_{\mathbf{vv}} & \Sigma_{\mathbf{vs}} \\
\Sigma_{\mathbf{sv}} & \Sigma_{\mathbf{ss}}
\end{bmatrix}, \quad (A.1)
$$

\textsuperscript{24} A pioneering model in this class is Kyle (1989).

\textsuperscript{25} This proof is due to Hamilton (1994). Projection theorem can be regarded as a special case of Kalman filter. For example, Ljungqvist and Sargent (2004) use that approach to prove the theorem.
where $\Sigma_{VV}$ is a $(n\times n)$ matrix, $\Sigma_{SS}$ is a $(m\times m)$ matrix, $\Sigma_{VS}$ is a $(n\times m)$ matrix and $\Sigma_{SV}$ is a $(m\times n)$ matrix. In order to obtain $0$ matrix in the upper right corner of matrix $\Sigma$, it is necessary to multiply this matrix with the following matrix:

$$E_1 = \begin{bmatrix} I_n & -\Sigma_{VS} \Sigma_{SS}^{-1} \\ 0 & I_m \end{bmatrix}. \quad (A.2)$$

By premultiplying matrix $\Sigma$ with matrix $E_1$ and postmultiplying matrix $\Sigma$ with matrix $E_1'$ we obtain diagonal block matrix $D$:

$$\begin{bmatrix} I_n & -\Sigma_{VS} \Sigma_{SS}^{-1} \\ 0 & I_m \end{bmatrix} \begin{bmatrix} \Sigma_{VV} & \Sigma_{VS} \\ \Sigma_{SV} & \Sigma_{SS} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -\Sigma_{SS}^{-1} \Sigma_{SV} & I_m \end{bmatrix} = \begin{bmatrix} \Sigma_{VV} - \Sigma_{VS} \Sigma_{SS}^{-1} \Sigma_{SV} & 0 \\ 0 & \Sigma_{SS} \end{bmatrix} = D. \quad (A.3)$$

Let matrix $A$ be defined as:

$$A \equiv E_1^{-1} = \begin{bmatrix} I_n & \Sigma_{VS} \Sigma_{SS}^{-1} \\ 0 & I_m \end{bmatrix}. \quad (A.4)$$

By premultiplying diagonal block matrix $D$ with matrix $A$ and postmultiplying matrix $D$ with matrix $A'$ we obtain matrix $\Sigma$:

$$\begin{bmatrix} \Sigma_{VV} & \Sigma_{VS} \\ \Sigma_{SV} & \Sigma_{SS} \end{bmatrix} = \begin{bmatrix} I_n & \Sigma_{VS} \Sigma_{SS}^{-1} \\ 0 & I_m \end{bmatrix} \begin{bmatrix} \Sigma_{VV} - \Sigma_{VS} \Sigma_{SS}^{-1} \Sigma_{SV} & 0 \\ 0 & \Sigma_{SS} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ \Sigma_{SS}^{-1} \Sigma_{SV} & I_m \end{bmatrix} = ADA'. \quad (A.5)$$

Thus, we have made a triangular factorization of matrix $\Sigma = ADA'$. From now on, we will assume that expected values of random vectors $v$ and $s$ are $E[v]$ and $E[s]$, respectively. Therefore, variance-covariance matrix becomes:

$$\Sigma = \begin{bmatrix} E(v - E[v])(v - E[v])' & E(v - E[v])(s - E[s])' \\ E(s - E[s])(v - E[v])' & E(s - E[s])(s - E[s])' \end{bmatrix} = \begin{bmatrix} \Sigma_{VV} & \Sigma_{VS} \\ \Sigma_{SV} & \Sigma_{SS} \end{bmatrix}. \quad (A.6)$$

Since random vectors $v$ and $s$ are normally distributed, joint density function is:
Thus, we have made a triangular factorization of matrix \( \Sigma \):

\[
\Sigma^{-1} = [ADA']^{-1} = [A']^{-1}D^{-1}A^{-1} = \begin{bmatrix} I_n & 0 \\ -\Sigma_{ss}^{-1}\Sigma_{sv} & I_m \end{bmatrix} \begin{bmatrix} (\Sigma_{vv} - \Sigma_{sv}\Sigma_{ss}^{-1}\Sigma_{sv})^{-1} & 0 \\ -\Sigma_{ss}^{-1} & 0 \end{bmatrix} \begin{bmatrix} I_n & -\Sigma_{vv}\Sigma_{ss}^{-1} \\ 0 & I_m \end{bmatrix}.
\] (A.8)

In the same fashion, we obtain the determinant of matrix \( \Sigma \):

\[
|\Sigma| = |A|\cdot|D|\cdot|A'|.
\] (A.9)

\( A \) and \( A' \) are triangular matrices with all elements equal 1 on the main diagonal. Hence, their determinant is 1, and it follows that \( |\Sigma| = |D| \):

\[
\begin{bmatrix} \Sigma_{vv} & \Sigma_{vs} \\ \Sigma_{sv} & \Sigma_{ss} \end{bmatrix} = \begin{bmatrix} \Sigma_{vv} - \Sigma_{sv}\Sigma_{ss}^{-1}\Sigma_{sv} & 0 \\ 0 & \Sigma_{ss} \end{bmatrix} = |\Sigma_{ss}| \begin{bmatrix} \Sigma_{vv} - \Sigma_{sv}\Sigma_{ss}^{-1}\Sigma_{sv} \end{bmatrix}.
\] (A.10)

By substituting (A8) and (A10) in the joint density function we obtain:

\[
g_{v,s}(v,s) = \frac{1}{(2\pi)^{(n+m)/2}} |\Sigma_{ss}|^{1/2} \begin{bmatrix} \Sigma_{vv} - \Sigma_{sv}\Sigma_{ss}^{-1}\Sigma_{sv} \end{bmatrix}^{1/2} \times \exp\left\{ -\frac{1}{2} [(v - E[v])' (s - E[s])] \begin{bmatrix} \Sigma_{vv} & \Sigma_{vs} \\ \Sigma_{sv} & \Sigma_{ss} \end{bmatrix}^{-1} \begin{bmatrix} (v - E[v]) \\ (s - E[s]) \end{bmatrix} \right\} \\
\times \begin{bmatrix} (\Sigma_{vv} - \Sigma_{sv}\Sigma_{ss}^{-1}\Sigma_{sv})^{-1} & 0 \\ 0 & \Sigma_{ss}^{-1} \end{bmatrix} \begin{bmatrix} I_n & -\Sigma_{vv}\Sigma_{ss}^{-1} \\ 0 & I_m \end{bmatrix} = \frac{1}{(2\pi)^{(n+m)/2}} |\Sigma_{ss}|^{1/2} \begin{bmatrix} (\Sigma_{vv} - \Sigma_{sv}\Sigma_{ss}^{-1}\Sigma_{sv})^{-1} & 0 \\ 0 & \Sigma_{ss}^{-1} \end{bmatrix} \begin{bmatrix} (v - m) \\ (s - E[s]) \end{bmatrix} \times \\
\begin{bmatrix} (\Sigma_{vv} - \Sigma_{sv}\Sigma_{ss}^{-1}\Sigma_{sv})^{-1} & 0 \\ 0 & \Sigma_{ss}^{-1} \end{bmatrix} \begin{bmatrix} (v - m) \\ (s - E[s]) \end{bmatrix}
\] (A.11)

where \( m = E[v] + \Sigma_{vs}\Sigma_{ss}^{-1}(s - E[s]) \).
We obtain conditional density of random vector $v$ with respect to a realization of random vector $s$ by dividing joint density function by the following marginal density function:

$$g_s(s) = \frac{1}{(2\pi)^{n/2}} |\Sigma_{ss}|^{-1/2} \exp\left\{-\frac{1}{2} (s - E[s])' \Sigma_{ss}^{-1} (s - E[s])\right\}.$$  \hspace{1cm} (A.12)

Hence, we obtain:

$$g_{vs}(v | s) = \frac{g_{vs}(v, s)}{g_s(s)} = \frac{1}{(2\pi)^{n/2}} |H|^{-1/2} \exp\left\{-\frac{1}{2} (v - m)' H^{-1} (v - m)\right\},$$  \hspace{1cm} (A.13)

where $H = \Sigma_{vv} - \Sigma_{vs} \Sigma_{ss}^{-1} \Sigma_{sv}$. Thus, we have proved the projection theorem:

$$v | s \sim N(m, H) \sim N\left[E[v] + \Sigma_{vs} \Sigma_{ss}^{-1} (s - E[s]), (\Sigma_{vv} - \Sigma_{vs} \Sigma_{ss}^{-1} \Sigma_{sv})\right].$$  \hspace{1cm} (A.14)

**B. Derivation of CARA-Gaussian demand function**

Let us assume that investor possesses a utility function $u(W) = -e^{-\rho W}$, where $\rho$ is his coefficient of absolute risk aversion and $W$ his wealth which is a normally distributed random variable. Investor maximizes the expected utility:

$$E[u(W) | \cdot] = \int_{-\infty}^{\infty} -e^{-\rho W} \cdot g(W | \cdot) \cdot dW,$$  \hspace{1cm} (B.1)

where $g(W | \cdot) = \frac{1}{\sqrt{2\pi} \sigma_w^2} \cdot e^{-\frac{(W - E[W])^2}{2 \sigma_w^2}}$. Substituting this result in (B.1) we obtain:

$$E[u(W) | \cdot] = \frac{1}{\sqrt{2\pi} \sigma_w^2} \cdot \int_{-\infty}^{\infty} -e^{-\rho W} \left[\frac{(W - E[W])^2}{2 \sigma_w^2}\right] \cdot dW = \frac{1}{\sqrt{2\pi} \sigma_w^2} \cdot \int_{-\infty}^{\infty} -e^{-\rho \left[\frac{Z}{2 \sigma_w^2}\right]} \cdot dW,$$  \hspace{1cm} (B.2)

where $Z = (W - E[W])^2 + 2 \cdot \rho \cdot \sigma_w^2 \cdot W$. By adding and subtracting $\rho^2 \cdot \sigma_w^4$ to the right hand side of the last equation we obtain $Z = (W - E[W])^2 + 2 \cdot \rho \cdot (E[W] - 0.5 \cdot \rho \cdot \sigma_w^2) \cdot \sigma_w^2$. If we employ this result in (B.2) we have:

---

26 This proof is due to Brunnermeier (2003). We have corrected some errors in his notes.
We obtain conditional density of random vector $v$ with respect to a realization of random vector $s$ by dividing joint density function by the following marginal density function:

$$
\pi = \frac{f(v, s)}{g(s)}
$$

Hence, we obtain:

$$
\pi = \frac{1}{\sqrt{2\pi \sigma_s^2}} e^{-\frac{1}{2\sigma_s^2} (s - E(s))^2} \quad \text{(A.12)}
$$

where

$$
\Sigma_s = H.
$$

Thus, we have proved the projection theorem:

$$
\pi = \frac{1}{\sqrt{2\pi \sigma_v^2}} e^{-\frac{1}{2\sigma_v^2} (v - E(v))^2}
$$

(B.14)

B. Derivation of CARA-Gaussian demand function

Let us assume that investor possesses a utility function

$$
u = \rho - W u(W)$$

where $\rho$ is his coefficient of absolute risk aversion and $W$ is his wealth which is a normally distributed random variable. Investor maximizes the expected utility:

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho - W u(W) \cdot dW
$$

(B.1)

where

$$
E(v) = \frac{1}{\sqrt{2\pi \sigma_v^2}} e^{-\frac{1}{2\sigma_v^2} (v - E(v))^2}
$$

(B.2)

Substituting this result in (B.1) we obtain:

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho - W u(W) \cdot dW
$$

(B.3)

The integral in (B.3) is equal to one, and maximizing expected utility is equivalent to maximizing the certainty equivalent:

$$
V(E[W], \sigma_w^2) = E[W] - 0.5 \cdot \rho \cdot \sigma_w^2.
$$

(B.4)

Let us now suppose that there are two kind of assets. Risk-free asset yields a return $r$, while fundamental value of the risky asset is $v \sim N(E[v], \sigma_v^2)$. Investor's initial endowments of risk-free and risky assets are $X_{F0}$ and $X_0$, respectively. His final demand of risk-free asset is $X_F$, and of risky asset $X$. Investor's budget constraint is $p \cdot X + X_F = p \cdot X_0 + X_{F0}$, where $p$ represents price of risky asset. Therefore, investor's final wealth is:

$$
W = X_F \cdot r + X \cdot v = (X_{F0} + p \cdot (X_0 - X)) \cdot r + X \cdot v.
$$

(B.5)

Expected value of final wealth is $E[W] = (X_{F0} + p \cdot (X_0 - X)) \cdot r + X \cdot E[v]$, and variance is $\text{var}[W] = \sigma_w^2 = X^2 \cdot \sigma_v^2$. Substituting $E[W]$ and $\sigma_w^2$ in (B.4) certainty equivalent becomes:

$$
V(E[v], \sigma_v^2) = E[W] - 0.5 \cdot \rho \cdot \sigma_v^2 = (X_{F0} + p \cdot X_0) \cdot r + X \cdot E[v] - pr - 0.5 \cdot \rho \cdot \sigma_v^2 \cdot X^2.
$$

(B.6)

Maximizing (B.6) with respect to $X$ we obtain:

$$
E[v] - p \cdot r - \rho \cdot \sigma_v^2 \cdot X = 0 \Rightarrow X(p) = \frac{E[v] - pr}{\rho \cdot \sigma_v^2}.\quad \text{■ (B.7)}
$$

---

27 Since $\rho \cdot \sigma_v^2$ is a constant, integrand in (B.3) represents distribution function of a normally distributed random variable $W$.  

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REFERENCES


