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## CONSUMPTION-BASED MACROECONOMIC MODELS OF ASSET PRICING THEORY

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**ABSTRACT:** *The family of consumption-based asset pricing models yields a stochastic discount factor proportional to the marginal rate of intertemporal substitution of consumption. In examining the empirical performance of this class of models, several puzzles are discovered. In this literature review we present the canonical model, the*

*corresponding empirical tests, and different extensions to this model that propose a resolution of these puzzles.*

**KEY WORDS:** *equity premium puzzle, stochastic discount factor, marginal rate of intertemporal substitution, risk-free rate puzzle, risk premium, volatility*

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## **1. INTRODUCTION**

Asset pricing theory follows two main streams. On the one hand there is the Equilibrium Pricing approach in which agents maximize their objective functions given their budget constraints, and after markets are cleared the equilibrium prices emerge. The primitives of this type of models are the joint distribution of the assets' payoffs and the agents' preferences. Since both of these are not easily visible in financial markets, the advantage of the second approach, the Arbitrage Pricing Theory, is that it does not require any knowledge of the preferences of the agents. This approach is based on the possibility of replicating the payoff of one asset by creating a portfolio of other assets that have already been priced. It follows that the price of such an asset should be equal to the price of the replicating portfolio.

Equilibrium pricing comprises many different models that postulate different preferences of agents. For example, the Capital Asset Pricing Model assumes that agents have preferences defined over risk and return so that the agents are the mean-variance optimizers. In this paper we are going to focus on the family of models in which agents have preferences over consumption and/or their wealth. Robert Lucas introduced a particular theoretical framework with a one-good, pure exchange economy with continuum of consumers that are identical in terms of their preferences and endowments (Lucas 1978), but the model can be extended to the economy that satisfies conditions which allow the existence of a representative agent. The agents decide how much to consume and how much to save, so that the pricing formula comes out as a result of their optimization. In this type of models, consumption is the key endogenous variable in the economy and the calibration requires looking at the macroeconomic variables.

The most important implications of these models are the pricing formulas that give the relation between the asset returns and exogenous variables, the interest rate formulas, and the mechanisms of the price corrections for risk. The Lucas 'Tree' model didn't do well in contrasting with the data. It predicts a lower equity risk premium and volatility than that observed in financial markets. It also predicts a higher risk-free rate than the one prevailing in the market. The main goal of this work is to provide readers with a comprehensive review of the

evolution of asset pricing literature inside this particular theoretical framework, and describe the puzzles that emerge from it.

The rest of the paper is organized as follows. The next section studies the baseline Lucas (1978) model of joint consumption and investment choice in a pure exchange economy and examines the equilibrium pricing equations. Section 3 presents the applications of the model, defines the most important asset pricing puzzles that arise from the consumption asset-pricing framework, and briefly describes the theoretical directions towards solving the puzzles. Section 4 concludes.

## 2. THE MODEL

### 2.1. Setup

The economy is populated by a continuum of homogeneous agents, but we can as well relax this assumption by allowing the agent heterogeneity under the conditions specified in Rubinstein (1974), which make aggregation into a representative agent possible. We consider a pure exchange economy, with one homogeneous good that serves as a numeraire.

Let  $(\Omega, \mathcal{F}, P)$  denote the probability space. There are  $n$  productive units that produce the good, such that at time  $t$  the productive unit  $i$  produces output  $y_{i,t}$ . Output vector at time  $t$  is given by  $y_t = (y_{1,t}, \dots, y_{n,t})$ . The state variable  $y_t$  is adapted to the filtration  $\{F_t\}_{t=0,1,\dots,\infty}$ . Production is exogenous and stochastic and it is assumed that the output vector process is a Markov chain with a time-invariant transition-conditional probability distribution function  $G : \mathbb{R}_+^n \times \mathbb{R}_+^n \rightarrow \mathbb{R}$ :

$$G(y_t, y) = P(y_{t+1} \leq y_t | y_t = y) \quad (1)$$

Assuming that  $y$  has a stationary distribution, the unconditional distribution is given by:

$$\varphi(y') = \int G(y', y) d\varphi(y). \quad (2)$$

Each productive unit has one perfectly divisible share traded at each time period  $t$ . The representative agent decides in each period  $t$  how much to consume and

what share of ownership of each productive unit to acquire. Buying a share of the productive unit today entitles the agent to the same share of the output of that productive unit in the next time period. Output is perishable, which means that it cannot be stored for consumption in later periods, which puts a boundary on the current consumption:

$$0 \leq c_t \leq \sum_{i=1}^n y_{t,i}. \quad (3)$$

The representative agent's preferences are described by the utility function  $U: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , which is bounded, continuous, at least twice differentiable, increasing, and strictly concave with  $U(0) = 0$ . He wishes to maximize the expected value of his utility function:

$$\mathbb{E}_0 \left[ \sum_{t=0}^{\infty} \beta^t \cdot U(c_t) \right], \quad 0 < \beta < 1 \quad (4)$$

By specifying the utility function in this way, we impose the time-separability of the consumer's preferences. The time preference is manifested only through the impatience factor  $\beta$ , and the utility derived from the consumption at period  $t$  does not depend on the consumption from any time period other than  $t$ . Also, with preferences defined as above, we do not have the dependence of utility of consumption on the particular state of the nature realization. This is the property of the state-independence of utility function.

Let  $z_t = (z_{1t}, \dots, z_{nt})$  be the vector of the agent's beginning-of-period share holdings at time  $t$ . If  $p_t$  is the prevailing price of one share of productive unit at time  $t$ , his budget constraint becomes:

$$c_t + p_t \cdot z_{t+1} = y_t \cdot z_t + p_t \cdot z_t. \quad (5)$$

At the beginning of each period  $t$ , the agent has  $z_t$  shares, which give him the wealth of  $(y_t + p_t) \cdot z_t$ . He earns the dividend  $y_t \cdot z_t$  and his share's market value is  $p_t \cdot z_t$ . The agent uses his wealth for consumption and for investing in shares, as a mean of transferring wealth from one period to the next.

## 2.2. Equilibrium

Before we give a formal definition of the equilibrium in this economy, we explain how the rational expectation equilibrium is established. One of the elements of equilibrium characterization is the equilibrium price, which is stochastic, given that it is a function of the state variable  $y_t$ . The quantity  $p(y_t)$  stands for the price of the production unit  $i$  when the economy is in the state  $y_t$ . By knowing the conditional probabilities of the economy changing from one state to another, we can fully specify the dynamic stochastic behaviour of the equilibrium price.

At time  $t$ , the agent has two choice variables,  $c_t$  and  $z_{t+1}$ . In making decisions he is taking into account how prices change over time as a function of the state of the economy. Equilibrium decision rules are  $c_t = c(y_t, z_t, p_t)$  and  $z_{t+1} = z(y_t, z_t, p_t)$ . These decision rules define how much the agent optimally consumes (invests in productive unit shares) at period  $t$ , given the state of the economy, the size of the ownership of production units at the beginning of the period, and the prevailing price. Given these decision rules, the market clearing conditions yield the equilibrium price function. In the Rational Expectation Equilibrium framework, the resulting price coincides with the price that the consumer conjectured when he was solving his optimization problem.

The formal definition of the equilibrium is based on the dynamic programming principle, where the representative agent's objective function is restated in a recursive form by writing the Bellman equation.

**Definition.** (Lucas 1978). An equilibrium is a continuous function  $p(y) : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$  and a continuous bounded function  $V(z, y) : \mathbb{R}_+^n \times \mathbb{R}_+^n \rightarrow \mathbb{R}$  such that:

$$V(z, y) = \max_{c, x} \{U(c) + \beta \cdot \mathbb{E}[V(x, y') | y_t = y]\} \quad (6)$$

such that

$$c + p(y) \cdot x \leq y \cdot z + p(y) \cdot z$$

$$c \geq 0$$

for  $\mathbf{0} \leq \mathbf{x} \leq \bar{\mathbf{x}}$ , where  $\bar{\mathbf{x}}$  is a vector with all the elements exceeding one and for each  $y$ ,  $V(\mathbf{c}, y)$  is attained by  $c = \sum_{i=1}^n y_i$  and  $\mathbf{x} = \mathbf{1}$ , where  $\mathbf{1}$  stands for the vector of ones.

The first part of the statement refers to the representative agent's optimal wealth allocation given the price behaviour. There is also a condition which does not allow negative consumption and the one which keeps the share positions finite given that the price may take a zero value. The last part refers to the market clearing conditions.

The non-negativity constraint on consumption can be omitted by assuming that Inada conditions hold. Apart from the concavity and monotonicity properties of the utility function that are already assumed, Inada conditions require that:

- $\lim_{c \rightarrow 0} U'(c) = +\infty$
- $\lim_{c \rightarrow \infty} U'(c) = 0$ .

Let us consider first the maximization part of the problem. Let  $W_t$  stand for the agent's wealth at time  $t$ , which comes from the market value of the share ownership at time  $t$  and the dividend  $y_t$  that is claimed by the agent. It is given by:

$$W_t = (y_t + p_t(y_t)) \cdot z_t \quad (7)$$

The time-separability of preferences allows the dynamic programming approach and using the Bellman equation method. Since the state variable  $y_t$  summarizes the investor's information set  $\mathcal{F}_t$ , we can use the notation  $J(W_t|y_t)$  for the value function instead of  $J(W_t|\mathcal{F}_t)$ .

$$J(W_t|y_t = y) = \max_{c_t, z_{t+1}} \left\{ U(c_t) + \beta \cdot \int J(W_{t+1}|y_{t+1} = y') dG(y', y) \right\} \quad (8)$$

such that

$$c_t + z_{t+1} \cdot p(y_t) = W_t.$$

Differentiating the Lagrangian function yields the first order conditions:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \frac{\partial U(c_t)}{\partial c_t} - \lambda_t = 0 \quad (9)$$

$$\frac{\partial \mathcal{L}}{\partial z_{t+1}} = \beta \cdot \int \frac{\partial J(W_{t+1} | y_{t+1} = y')}{\partial W_{t+1}} \cdot \frac{\partial W_{t+1}}{\partial z_{t+1}} dG(y', y) - \lambda_t \cdot p(y_t) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = W_t - c_t - z_{t+1} \cdot p(y_t) = 0$$

We write only the price as an explicit function of the state, even though the consumption  $c_t$ , the share positions  $z_t$  and the shadow prices  $\lambda_t$  are also functions of the state, because the investor makes his optimal decisions contingent on the state of the economy. Since  $W_{t+1} = z_{t+1} \cdot (p(y_{t+1}) + y_{t+1})$ , the partial derivative of the wealth in the next period with respect to the share position the investor buys today is equal to:

$$\frac{\partial W_{t+1}}{\partial z_{t+1}} = p(y_{t+1}) + y_{t+1} \quad (10)$$

Notice that in order to make  $W_t$  an explicit argument of the value function  $J$ , we substitute  $c_t = W_t - z_{t+1} \cdot p(y_t)$  from the constraint into the utility function part of the objective  $J$ . From there it follows that  $\frac{\partial J(W_t | y_t = y)}{\partial W_t} = \frac{\partial U(c_t)}{\partial c_t}$  and it also holds for the partials from the next period. Since value function  $J$  represents the total discounted utility, given that the investor made the optimal consumption and investment decisions, this condition requires that the marginal total utility of current wealth equals the marginal current utility of consumption. The marginal utility of one unit of consumption good spent today must be the same as the marginal value of one unit of consumption good saved and spent some time in the future. The investor has to be indifferent between purchasing the tree and selling the tree and using the proceedings for the current consumption.

Using the equality between the marginal utility and the marginal value to rewrite the first order conditions and combining them yields the fundamental pricing equation:

$$p(y_t) = \frac{\beta \cdot \frac{\partial U(c_{t+1})}{\partial c_{t+1}} \cdot (p(y_{t+1}) + y_{t+1}) \cdot dG(y', y)}{\frac{\partial U(c_t)}{\partial c_t}} \quad (11)$$

Mathematically, it is an Euler equation. The economic interpretation of this equation is that the market price of an asset today is equal to the expected value of the asset's payoff tomorrow, adjusted by the ratio of marginal utilities of consumption tomorrow and today. When marginal utility of consumption tomorrow is low, it means that the consumption is high and that the state's payoff will be assigned lower weight in the pricing equation. It is the opposite for the states with low consumption, where marginal utility of the additional dividend is more valuable.

The increase in expected output increases the attractiveness of holding the share of the asset, which pushes the price of the asset up. However, higher expected output immediately means higher expected consumption, because the consumption good is not storable and the agent consumes the entire output in each state. If the expected consumption is high the marginal utility is low, which has a negative effect on the asset price. What the resulting effect of these two is depends on the particular shape of the utility function. In this model the consumption good cannot be stored, so the equilibrium price of the asset is adjusted to the point at which the investor consumes the entire dividend in each period, which implies that we can substitute  $c_t$  for  $y_t$  as the argument of the marginal utility function in the equation (11).

The alternative way of expressing the Euler equation is to introduce pricing kernel or stochastic discount factor that measures the willingness of the investor to shift consumption between two dates and is equal to:

$$m_{t,t+1}(y', y) = \beta \cdot \frac{\frac{\partial U(y_{t+1})}{\partial y_{t+1}}}{\frac{\partial U(y_t)}{\partial y_t}} \quad (12)$$

It is defined as a product of the marginal rate of intertemporal substitution and the subjective discount factor. Substituting it into the pricing equation (11) and expressing it in terms of the expected value<sup>1</sup> yields:

$$p(y_t) = E[m_{t,t+1}(y', y) \cdot (p(y_{t+1}) + y_{t+1}) | y_t = y] \quad (13)$$

The economic interpretation of this equation is that the price of the asset is the expected value of the discounted value of future payoffs, with the distinction that the payoffs are transformed into ‘utils’ by using the stochastic discount factor.

The utility function is strictly increasing and the subjective discount factor is strictly positive, which ensures that the stochastic discount factor is strictly positive in each state of the world. This is an important property of the stochastic discount factor, because it prevents the presence of arbitrage opportunities in the economy. No arbitrage condition is stronger than the requirement that the Law of One Price (LOOP) holds. We do not need the stochastic discount factor to be positive in order for the financial markets to obey LOOP, but only to have a linear pricing mechanism, which is the condition equivalent to the mere existence of the stochastic discount factor. For detailed proof of equivalence between LOOP and the linear pricing mechanism on the one hand and the no arbitrage condition and strictly positive stochastic discount factor on the other, see Cochrane (2005) or Duffie (2001).

We can also express the pricing formula in terms of returns on assets. If we define the gross return on holding the risky asset between periods  $t$  and  $t + 1$  as:

$$R_{t,t+1} = \frac{p(y_{t+1}) + y_{t+1}}{p(y_t)} \quad (14)$$

then the pricing formula is given by:

$$1 = E_t[m_{t,t+1}(y', y) \cdot R_{t,t+1}] \quad (15)$$

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<sup>1</sup> In the rest of the text, the expected value conditional on the information from the state  $y_t = y$  will be denoted using the subscript  $t$  next to the expectation operator.

Now we turn to the Consumption Capital Asset Pricing Model derivation. We are going to relax the assumption of the non-existence of the consumption good storage technology. With this assumption, the optimal investor's behaviour is to consume the entire output. In that way the marginal rate of substitution and the stochastic discount factor are both determined by the exogenous fluctuations in the aggregate output. With the possibility of saving the output for future periods, the investor's consumption does not coincide with the aggregate output in each state over the entire investment horizon. Consumption  $c_t$  is replaced in equation (12) and the stochastic discount factor's distribution is derived from the distribution of the investor's consumption.

First, we analyse the implications for the riskless asset, the one which has certain constant payoff in every state of the world. That means that  $R_{t,t+1}^f$  is a constant and can be taken out of the expectation operator in equation (15). That gives us the expression for the riskless return in this economy:

$$R_{t,t+1}^f = \frac{1}{E_t[m_{t,t+1}(y',y)]} \quad (16)$$

The riskless gross return is just the reciprocal of the expected value of the stochastic discount factor.

We now go back to the risky asset by using the Euler equation expressed in terms of the gross returns as in the equation (15), and use the product rule to obtain:

$$E_t[R_{t,t+1}] - R_{t,t+1}^f = -\frac{1}{R_{t,t+1}^f} \cdot Cov_t[m_{t,t+1}, R_{t,t+1}] \quad (17)$$

This is the Consumption Capital Asset Pricing Model (CCAPM); for the continuous time version see Breeden (1979). From this transformation of the Euler equation we can learn something about the pricing relation. The higher the covariance between the asset's payoff and the marginal rate of substitution, the more expensive the asset and the lower the risk premium required by the investor because it provides good insurance against undesired changes in consumption. If the marginal rate of substitution is high it means that the consumption level is low, so that the asset with high payoff in that state is more

valuable to the investor. An asset that covaries positively with consumption is poorly appreciated compared to the one that covaries negatively. This model already provides the intuition that what matters is not the ‘individual’ but the ‘common’ risk in pricing the assets.

### 2.3. Asset Prices Without Bubbles

In order to prevent the possibility of creating a pricing bubble, the investor’s maximization problem must satisfy the following transversality condition:

$$\lim_{k \rightarrow \infty} E_t \left[ \beta^k \cdot \frac{\frac{\partial U(c_{t+k})}{\partial c_{t+k}}}{\frac{\partial U(c_t)}{\partial c_t}} \cdot p(y_{t+k}) \right] = 0 \quad (18)$$

In order to see why this condition is important, we perform recursions on the asset pricing equation (13), relying on the law of iterated expectations, which yields:

$$p(y_t) = E_t \left[ \sum_{j=1}^{\infty} \beta^k \cdot \frac{\frac{\partial U(c_{t+j})}{\partial c_{t+j}}}{\frac{\partial U(c_t)}{\partial c_t}} \cdot y_{t+j} \right] + \lim_{k \rightarrow \infty} E_t \left[ \beta^k \cdot \frac{\frac{\partial U(c_{t+k})}{\partial c_{t+k}}}{\frac{\partial U(c_t)}{\partial c_t}} \cdot p(y_{t+k}) \right] \quad (19)$$

If the transversality condition (18) holds, we have that the price of the asset is equal to the expected value of the discounted dividend stream, but with the time varying and stochastic discount factors.

$$p(y_t) = E_t \left[ \sum_{j=1}^{\infty} \beta^k \cdot \frac{\frac{\partial U(c_{t+j})}{\partial c_{t+j}}}{\frac{\partial U(c_t)}{\partial c_t}} \cdot y_{t+j} \right] \quad (20)$$

To know more about the risky asset’s pricing mechanism we need to introduce more structure into the model. In particular, certain assumptions need to be made about the probability distribution of the aggregate output, the investor’s preferences, and the market structure. For example, under a very restrictive

assumption about the investor's risk preferences, such as risk neutrality, the Euler equation can collapse to the equilibrium price of the risky asset being equal to the expected discounted value of the asset's dividend stream with constant discount factor. The risk-neutral investor is endowed with a linear utility function, which implies that the marginal utility is constant and the marginal rate of substitution is equal to one. The covariance term in (17) is equal to zero and the implied pricing formula is:

$$p(y_t) = E_t[\beta \cdot (p(y_{t+1}) + y_{t+1})] \quad (21)$$

In the next section we introduce more realistic assumptions regarding the investor's preferences and discuss the implications of the CCAPM application.

### **3. APPLICATION OF THE LUCAS MODEL**

In this section we present Mehra and Prescott's work on testing the Lucas model using U.S. data on aggregate consumption, stock market returns, and returns on government bonds. They show that this class of models cannot generate an average annual equity premium of more than 6% without assuming an unrealistically high value of the risk-aversion parameter. Prior to their work, Hansen and Singleton (1982, 1983) used both the Generalized Method of Moments with instrumental variables and the Maximum Likelihood Estimation method on a canonical form of consumption-based model. They tested the model on a time series of US monthly data on consumption growth and asset returns, to find that the consumption-based pricing model is rejected. We choose to present the Mehra and Prescott approach in greater detail, as they introduce additional structure into the Lucas model and, in so doing, produce a very simple test of the model's empirical performance.

Beside the equity premium puzzle, we also present the risk-free rate puzzle identified within this framework, and another test of the model validity which requires much less of the model structure.

#### **3.1. Equity Premium Puzzle**

In the paper called "The Equity Premium: A Puzzle", Mehra and Prescott (1985) consider a pure exchange economy with one consumption good produced by

one productive unit. They make a slight variation in the model by changing several assumptions. First, they observe that the U.S. per capita consumption level exhibited a large increase over the period 1889-1978 that does not allow assuming stationarity of the consumption process, as Lucas did. Instead of assuming that the total output, with productivity exogenously given, follows the Markov process, they assume that the growth rate of the aggregate output is a Markov process. Second, they discretize the setup in terms of states. In each period of time there is a finite number of states, described by a grid of values that the dividend growth rate can take. In addition, they put structure on the investor's preferences by assuming the power utility function. For the sake of consistency, we do not follow the paper closely and preserve the assumption of a continuous-state economy. The results remain the same.

The economy is described as a pure exchange one, with one consumption good produced by only one productive unit, i.e., there is only one Lucas tree in the economy. Only one share of equity of the productive unit is traded, which implies that the return on that asset is, at the same time, the return on the market. Uncertainty in the economy is described by the probability space  $(\Omega, F, P)$ . The output produced by the productive unit  $y_t$ , which represents the dividend received from the ownership over the productive unit, is exogenous and stochastic with the growth rate being conditionally normally distributed with mean  $\mu_t$  and variance  $\sigma_t$ , given the information available up to time  $t$ . We define the growth rate as:

$$\Delta \ln y_{t+1} = \ln \frac{y_{t+1}}{y_t} \quad (22)$$

The investor has the power utility function:

$$U(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}, \quad \gamma \geq 0 \quad (23)$$

Note that the log utility is a special case of power utility function for relative risk aversion (RRA) parameter  $\gamma=1$ . Time preference parameter  $\beta$  is given by the expression  $e^{-\theta}$  such that  $\theta > 0$ . Given the particular shape of the investor's preferences and the expression for the time preference parameter, combining

them into the Euler equation for asset returns produces the following pricing equation:

$$\mathbf{E}_t \left[ e^{-\theta} \cdot \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \cdot R_{t,t+1} \right] = \mathbf{1} \quad (24)$$

First we derive the implications for the risk-free rate between periods  $t$  and  $t+1$ . Since  $R_{t,t+1}^f$  is not random but a constant, it follows that the gross risk-free return is equal to:

$$R_{t,t+1}^f = \frac{\mathbf{1}}{\mathbf{E}_t \left[ e^{-\theta} \cdot \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \right]} \quad (25)$$

Since the representative investor consumes the entire output, the consumption growth is conditionally normally distributed identically as the output growth, and it follows that:

$$\frac{\mathbf{1}}{R_{t,t+1}^f} = \mathbf{E}_t \left[ e^{-\theta} \cdot e^{-\gamma \Delta \ln c_{t+1}} \right] = e^{-\theta} \cdot e^{-\gamma \cdot \mu_t - \frac{\gamma^2}{2} \cdot \sigma_t^2} \quad (26)$$

Let  $r_{t,t+1}^f$  be the net risk-free return equal to  $\ln R_{t,t+1}^f$ . By taking the natural logarithm of the inverse of the expression above we obtain:

$$r_{t,t+1}^f = \theta + \gamma \cdot \mu_t - \frac{\gamma^2}{2} \cdot \sigma_t^2 \quad (27)$$

Let us look at the comparative statics of expression (27):

- The higher the impatience factor  $\theta$  the higher the risk-free rate, because the investor prefers to consume early; which is manifested in him reducing his savings, thus having the risk-free rate increasing in the equilibrium.
- When the consumer has a concave utility function, he wants to smooth his consumption over time (and over states as well). The higher the expected

consumption growth rate  $\mu_c$  the more he borrows today against future consumption, and the equilibrium risk-free rate goes up.

- High consumption growth volatility  $\sigma_c^2$  makes the future consumption more uncertain, which leads to the investor saving more, for precautionary motives. As a consequence, in the equilibrium the risk-free return drops.
- The risk aversion parameter  $\gamma$  magnifies the effect of change in the expected growth rate of consumption and the consumption growth volatility. The more risk-averse the investor, the more he tends to smooth the consumption over time, but whether he borrows or saves more today depends on the particular values of parameters  $\mu_c$  and  $\sigma_c^2$ .

Now, let us turn to the risky asset with the gross return being a random variable  $R_{t+1}$ . It is assumed that  $r_{t+1} = \ln R_{t+1}$  is normally distributed, conditional on the information from time  $t$ , with the mean  $E_t[r_{t+1}]$  and the variance  $Var_t[r_{t+1}]$ . We start with the Euler equation by expressing the consumption growth and the return  $R_{t+1}$  in exponential terms:

$$E_t \left[ e^{-\delta - \gamma \cdot \Delta \ln c_{t+1} + r_{t+1}} \right] = 1 \quad (28)$$

Calculating the expectation, taking the natural logarithm of both sides and combining it with the expression for the risk-free rate yields:

$$-r_{t+1}^f + E_t[r_{t+1}] + \frac{1}{2} \cdot Var_t[r_{t+1}] - \gamma \cdot Cov_t[\Delta \ln c_{t+1}, r_{t+1}] = 0 \quad (29)$$

In order to simplify the expression above we use the following two approximations:

$$E_t[R_{t+1}] = e^{E_t[r_{t+1}] + \frac{1}{2} Var_t[r_{t+1}]} \approx 1 + E_t[r_{t+1}] + \frac{1}{2} Var_t[r_{t+1}] \quad (30)$$

and

$$R_{t+1}^f = 1 + r_{t+1}^f \quad (31)$$

In terms of the equity premium, the pricing relation is given by:

$$E_t[R_{t+1}] - R_{t+1}^f = \gamma \cdot \text{Cov}_t[\Delta \ln c_{t+1}, r_{t+1}] \tag{32}$$

In terms of the Sharpe ratio (equity premium per unit of risk), the pricing formula becomes:

$$\frac{E_t[R_{t+1}] - R_{t+1}^f}{\sigma_t(r_{t+1})} = \gamma \cdot \text{Corr}_t[\Delta \ln c_{t+1}, r_{t+1}] \cdot \sigma_t(\Delta \ln c_{t+1}) \tag{33}$$

If we look at the equations (32) and (33), we see that the price of the risk is governed by:

- The risk aversion parameter  $\gamma$ , so that the higher the willingness of the investor to smooth the consumption the higher the risk premium required.
- The higher the correlation between the risky asset payoff and the consumption growth the higher the discount at which the risky asset is traded. If the economy is in a bad state, meaning that consumption growth is small, the asset which pays off a lot in such a state is expensive compared to the one that pays off a lot in a good state and the equity premium is small.
- The higher the consumption volatility  $\sigma_t(\Delta \ln c_{t+1})$ , the higher the price of the risk, since holding the risky asset makes the consumption even more unbalanced.

**Table 1.** Mehra and Prescott (1985) U.S. Economy Sample Statistics (1889–1978)

Statistic	Value
Risk-free rate $r_{t+1}^f$	0.008
Mean return on equity $E_t[r_{t+1}]$	0.0698
Stock market volatility $\sigma_t(r_{t+1})$	0.1667
Mean growth rate of consumption $\mu_t$	0.018
Consumption growth rate volatility $\sigma_t$	0.036
Mean equity premium $E_t[r_{t+1}] - r_{t+1}^f$	0.0618

In order to test the validity of the Lucas model predictions, Mehra and Prescott calibrated it with data on the US stock market and the aggregate consumption. In Table 1 we report the sample statistics for the US economy as used in Mehra

and Prescott (1985). Plugging these values into the expression for the equity premium (32) and assuming that the correlation coefficient between consumption growth and market return is equal to one (which ceteris paribus gives the maximum equity premium) yields:

$$E_t[r_{t+1}] - r_{t+1}^f = \gamma \cdot \sigma_t \cdot \sigma_t(r_{t+1}) = \gamma \cdot 0.006001 \quad (34)$$

We need an excessively large relative-risk-aversion coefficient  $\gamma$  of at least 10.3 in order to rationalize the equity premium of 6.18 as evidenced on the stock market. A number of studies reported in Mehra and Prescott (1985) found that the risk aversion coefficient takes a value between one and two. The value that is typically used in the literature is three. This discrepancy between the model predictions and the stock market evidence is called the equity premium puzzle. The equity premium puzzle is a puzzle of a quantitative nature. The models described above are correct in predicting that the risky assets should on average yield higher returns than bonds because of the higher risk attached. However, they fail to predict the order of magnitude of the difference in returns, compared to that which has been historically documented.

### 3.2. Risk-free Rate Puzzle

Philippe Weil (1989) invented the term ‘risk-free rate puzzle’ in an attempt to resolve the equity premium puzzle by leaving the time-additive expected utility framework and introducing the generalized non-expected utility specification, which allows breaking the link between intertemporal elasticity of substitution and risk aversion parameter, which are both functions of the curvature of the utility function, and in the time-additive framework represent the inverse of each other. Separating the risk aversion from the intertemporal substitution explains the equity premium but risk-free rate puzzle emerges, because the parameters that fit the equity premium yield the risk-free rate far higher than the 0.8% observed in reality. In this paper we will expose the problem of the risk-free rate puzzle in the framework of the Lucas model, i.e., we are going to stay within the theoretical domain of the expected utility. If we plug the values from Table 1 into the equilibrium expression for the risk-free rate as given in equation (27), and if we take the parameter  $\delta$  to be equal to 0.01005 so that the subjective discount rate  $\beta = e^{-\delta} = 0.99$ , then the risk-free rate is equal to:

$$r_{t+1}^f = 0.01005 + \gamma \cdot 0.018 - \frac{\gamma^2}{2} \cdot 0.036$$

The risk-aversion parameter that is used to rationalize the historical equity premium is higher than 10. If we plug this value into the equation above we obtain a risk-free rate equal to 12.52%, which is far above 0.8%, the risk-free rate that characterized the US economy over the period 1889-1978.

### 3.3. Hansen-Jagannathan Bounds

In their completely non-parametric approach, which uses the minimum structure possible, Hansen and Jagannathan (1991) find the admissible region for the expected value and standard deviation of the stochastic discount factor. In order to derive the lower bound of the volatility of the stochastic discount factor, the only requirement is that the Law of One Price holds in financial markets, which means that the stochastic discount factor exists as well as the absence of arbitrage opportunities, which requires that the stochastic discount factor is positive. For this purpose, we do not even need markets to be complete.

For the sake of simplicity, they focus on the unconditional version of the Euler equation and use the risk-free return as a varying parameter, which is equal to the inverse of the mean stochastic discount factor, in order to trace out the lower bound of the volatility of SDF. Conditioning down pricing equations creates a problem for empirical financial economists, since in reality it is observed that the conditional moments of return and SDF distributions vary over time because investors acquire new information. However, for the sake of consistency we will present their result using the equations with conditional moments.

Hansen and Jagannathan start from the general pricing equation (15), apply it to the risk-free and the risky asset, and take the difference between the two to get:

$$E_t[m_{t+1}(R_{t+1} - R_{t+1}^f)] = 0 \quad (35)$$

By the product rule, the left hand side can be rewritten as  $E_t[m_{t+1}] \cdot E_t[R_{t+1} - R_{t+1}^f] + Cov_t[m_{t+1}, R_{t+1} - R_{t+1}^f]$ . Expressing the covariance term

as a function of the correlation coefficient and after some simple algebraic manipulation it follows that:

$$\frac{E_t[R_{t+1}] - R_{t+1}^f}{\sigma_{R_{t+1}}} = -\rho_t(m_{t+1}, R_{t+1}) \cdot \sigma_m \cdot R_{t+1}^f \quad (36)$$

Since the absolute value of the correlation coefficient is bounded to be below or equal to one, we get the lower bound for the volatility of the stochastic discount factor:

$$\sigma_m \geq \frac{E_t[R_{t+1}] - R_{t+1}^f}{\sigma_{R_{t+1}} \cdot R_{t+1}^f} \quad (37)$$

This expression states that the lower bound for the stochastic discount factor has to be at least equal to the Sharpe ratio of the stock market divided by the gross risk-free rate. This bound serves as a test for any proposed asset pricing model.

If we go back to the Mehra and Prescott's stock market statistics in Table 1, we can test the consumption-based asset pricing model derived under the additional assumptions in Section 3. We get that:

$$\sigma_m \geq \frac{1.0698 - 1.008}{0.1667 \cdot 1.008} = 0.3677.$$

Since the stochastic discount factor's volatility in the consumption-based asset pricing model is equal to the product of the volatility of the consumption growth rate that is equal to 0.036 and the relative risk aversion coefficient  $\gamma$ , it requires the RRA coefficient to have a value of at least 10. As before, since this is an unrealistically high coefficient value the consumption-based model did not perform well.

### 3.4. Model Extensions

The Lucas model has been extended in various directions in order to improve its empirical performance. The two most common developments have been around

different specification of the utility function and introducing agent heterogeneity into the model.

The models presented in this paper use the standard preference representation characterized by time-separability, which implies that the coefficients of elasticity of intertemporal substitution and the relative risk aversion are the reciprocals of each other. Epstein and Zin (1989) introduced the class of functions that describe the recursive preferences. The most important characteristic is that the link between the two abovementioned coefficients is broken and the risk-free rate puzzle can be mitigated, because one can achieve higher expected market return by increasing the RRA coefficient, for the purpose of solving the equity premium puzzle, without implying an unrealistically high risk-free rate as a consequence.

An alternative class of models keeps the assumption of time-separability of preferences, but assumes that the agent's utility depends on a certain subsistence level of consumption that serves as a benchmark. If the actual consumption level drops below the subsistence level, it yields negative utility to the agent. This stream of the literature was pioneered by Sundaresan (1989), Constantinides (1990), and Abel (1990). The further advancement was made by Campbell and Cochrane (1999). The basic idea is that the agents have formed a certain level of habit which results in the effective risk aversion of a larger scale for the agent, thus helping in the consumption-based model fitting the historical value of the equity premium.

Models with heterogeneous agents (Mankiw 1986, Constantinides et al. 1996) typically assume that the agents have individual income shocks that limit the possibility of risk-sharing. With increasing cross-sectional volatility of consumption in states where the asset returns are low, the assets' risk premium increases beyond the level yielded in the model with a representative agent. Krueger and Lustig (2009) show that in an economy with a continuum of agents the risk premium is not affected by the absence of an insurance market if the aggregate shocks are distributed independently from the individual shocks, and if the consumption growth is independently distributed over time.

Other attempts to solve the financial puzzles go in the direction of specifying alternative economics processes, such as introducing a persistent long-run

component in return and consumption growth processes (Bansal et al. 2004), or in the direction of studying the effect of different market frictions on the market risk premium. For the latter, see He et al. (2005).

#### 4. CONCLUSION

The consumption-based general equilibrium class of models provides a pricing formula with very sharp intuition. The assets are priced in such a way that buying an asset at a certain price incurs consequent marginal loss in utility from current consumption equal to the discounted expected gain in utility of consuming the asset's payoff in the future. Even though this class of model is correct in explaining the sign and the direction of the effects of one financial variable on another, it fails in predicting the magnitude of the average return on the stock market or the level and variation of interest rates. In terms of the power of predicting the variables of interest, these models can be improved by an alternative specification of agent's preferences or by alternative assumptions of different economic processes. The two most commonly used approaches are to change the assumptions regarding the agent's utility functions and to assume the heterogeneity among agents.

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