ABSTRACT: The concept of Value at Risk (VaR) estimates the maximum loss of a financial position at a given time for a given probability. This paper considers the adequacy of the methods that are the basis of extreme value theory in the Montenegrin emerging market before and during the global financial crisis. In particular, the purpose of the paper is to investigate whether the peaks-over-threshold method outperforms the block maxima method in evaluation of Value at Risk in emerging stock markets such as the Montenegrin market. The daily return of the Montenegrin stock market index MONEX20 is analysed for the period January 2004 – February 2014. Results of the Kupiec test show that the peaks-over-threshold method is significantly better than the block maxima method, but both methods fail to pass the Christoffersen independence test and joint test due to the lack of accuracy in exception clustering when measuring Value at Risk. Although better, the peaks-over-threshold method still cannot be treated as an accurate VaR model for the Montenegrin frontier stock market.

KEY WORDS: Extreme value theory, Value at Risk, fat tails, Block maxima method, Peaks over threshold method, Generalized Pareto distribution.

JEL CLASSIFICATION: C13, C22, G10
1. INTRODUCTION

During the global financial turmoil, assessing and modelling the behaviour of extreme events has gained importance in economics. The methodology used for the assessment of financial market participants’ rate of exposure to risk gives the estimation of Value at Risk (VaR). Value at Risk is the maximum loss of a financial position over a given time period at a given confidence interval.

J P Morgan introduced VaR in 1994 as a method of risk management. Although it is established as an actual and central risk measure in regulatory frameworks, it is often criticized for its many shortcomings. First of all, VaR is not a coherent risk measure because it undermines the property of sub-additivity (Artzner et al. 1997).

Extreme value theory is a powerful tool that is being increasingly used in VaR estimation. In extreme value theory there are two approaches to measuring extreme values: 1) The block maxima method, which considers extreme observations from successive periods called blocks, and 2) the peaks-over-threshold method, which focuses on observations that exceed a particular threshold. The aim of this paper is to present an assessment of VaR using these two methods to measure Value at Risk in the still-developing Montenegrin financial market. The results will be compared and tested in order to determine which application most efficiently uses data on extreme values. The backtesting procedure, based on the results of the Kupiec test and Christoffersen’s (1998) independence and joint tests, is used to evaluate the VaR estimates calculated by these methods.

The purpose of this paper is to use extreme value theory to evaluate VaR in the Montenegrin stock exchange over a long period that includes years of financial crisis. In particular, we investigate whether the peaks-over-threshold method can outperform the block maxima method in the calculation of VaR in emerging stock markets, especially as the Montenegrin stock market has not been discussed in empirical literature until recently (Cerovic 2014). The Montenegrin financial market is a typical frontier market, a term used to describe emerging markets whose financial system in general, and stock exchange in particular, exhibit a lesser degree of development than traditional, long-standing emerging markets (De Groot et al. 2012). The main problems of
frontier markets are high concentration, low trading volumes, relatively inexperienced professional investors, incomplete institutional design, and low transparency. Our contribution is to include the Montenegrin market in risk management literature, as due to a lack of data it has not been analysed until now - as is the case for all emerging markets (Mladenovic et al. 2012). Given that there is less information generally available regarding frontier markets, our results are particularly useful to professional investors, whose interest in frontier markets has grown over recent years. Another contribution of this paper is to test the performance of VaR estimated by blocks of maxima and peaks-over-threshold on the Montenegrin stock market and to analyse whether there is a significant difference in the results obtained.

The paper is organized as follows. A literature review is presented in the next section. The third section reviews the basic concepts of the two methods used in extreme value theory, block maxima and peaks-over-threshold. We also present the backtesting procedure. Data used in the study and descriptive statistics are presented in fourth section. The fifth section presents the empirical analyses and comparison of these two methods using the backtesting procedure. Finally, conclusions are presented in the sixth section.

2. LITERATURE REVIEW


There are very few papers that compare VaR models that include extreme value theory in emerging financial markets. Da Silva and Mendes (2003) investigated VaR estimates in Asian emerging markets using extreme value theory and then compared the results with empirical and normal estimates. The results suggested that the extreme value method of estimating VaR is a more conservative approach to determining capital requirements than traditional methods. Gençay and Selçuk (2004) analysed parameter models and quantile assessment of the VaR of stock exchange indices in developing Central and
Eastern European countries. The results showed that generalized Pareto distribution and extreme value theory are basic tools in risk management in developing countries. Bao, Lee, and Saltoglu (2006) examined the stock markets of five Asian economies (Indonesia, Malaysia, Korea, Taiwan, Thailand) during the financial crisis and showed that some EVT-based models did better in crisis periods. Maghyereh and Al-Zoubi (2006) showed that the EVT approach is superior to classical variance-covariance and historical simulation for seven Middle Eastern and North African countries. Cotter (2004, 2007) found that EVT estimates for Value at Risk and expected shortfall outperform other estimates, analysing six Asian markets during the Asian crisis and five equity indices from European markets. Zikovic and Aktan (2009) analysed VaR models of the returns of Turkish and Croatian stock-exchange indices with the onset of the global financial crisis. The paper concluded that extreme value theory and hybrid historical simulation are best, while other models underestimate the level of risk. Assaf (2009) and Andreev et al. (2010) used the conditional generalized Pareto distribution to successfully model risks in emerging markets, the equity markets of the Middle East and North Africa (MENA) region in the former case and the Russian stock market in the latter. Andjelic, Milosev, and Djakovic (2010) investigated the performance of extreme value theory on the daily stock index returns of four different emerging markets (Serbia, Croatia, Slovenia, and Hungary), and concluded that the EVT approach should include continuous monitoring, with special emphasis on the role of optimal threshold determination. Nikolic-Djoric and Djoric (2011) observed the movement of the stock exchange index in the Serbian financial market and concluded that the generalized autoregressive conditional heteroscedasticity (GARCH) models combined with extreme value theory – the peaks-over-threshold method - decrease the mean value of VaR, and that those models are better than the RiskMetrics method and the integrated GARCH (IGARCH) model. Based on analysis of stock exchange indices in Central and Eastern European countries (Bulgaria, Czech Republic, Hungary, Croatia, Romania, and Serbia), Mladenovic, Miletic, and Miletic (2012) came to the conclusion that the methodology of extreme value theory is slightly better than the GARCH model regarding the calculation of VaR, but suggested that both approaches be used to better measure market risk.
3. METHODOLOGY

In order to determine the level of risk over the period \([t, t+h]\) we observe a portfolio of risky assets and determine the portfolio value as \(V_t\) at moment in time \(t\). Then the random variable of portfolio loss is \(L_{t+h} = -(V_{t+h} - V_t) = \Delta V(h)\). The cumulative function of loss distribution is marked as \(F_L\), where \(F_L(x) = P(L \leq x)\). In this case, VaR at significance level \(\alpha\) (most often \(\alpha = 0.01\) or \(\alpha = 0.05\)) is actually an \(\alpha\)-quantile of distribution function \(F_L\) and represents the smallest real number satisfying the inequation \(F_L(x) \geq \alpha\), i.e.,

\[
VaR_\alpha = \inf(x | F_L(x) \geq \alpha) .
\]  

(1)

Extreme value theory studies the tail behaviour of a distribution. Let \(X_1, X_2, \ldots\) be the series of independent, non-degenerate random variables having identical distribution, with the joint distribution function \(F\). Let us observe the maximum values of variables \((M_1 = X_1)\)

\[
M_n = \max(X_1, \ldots, X_n) .
\]  

(2)

The joint limiting distribution function of maxima \(M_n\), based on the character of their independence, is:

\[
P(M_n \leq x) = P(X_1 \leq x, \ldots, X_n \leq x) = \prod_{i=1}^{n} P(X_i \leq x) = \prod_{i=1}^{n} F(x) = F^n(x) .
\]  

(3)

The problem is to determine real constants \(a_n > 0\) and \(b_n\), so the variable \(\frac{M_n - b_n}{a_n}\) has a non-degenerate limiting distribution, as \(n \to \infty\), i.e.,

\[
\lim_{n \to \infty} F^n(a_n x + b_n) = G(x) .
\]  

\(G\) represents the non-degenerate distribution function. Such distributions are called extreme value distributions, and \(G\) then belongs to one of three distributions:
These block maxima data are used in parameter evaluation of generalized extreme value distribution. It is logical to conclude that this estimation will be influenced by block size \( n \).

Assuming the block maxima fit generalized extreme value distribution, the probability density function of standardized variables is given as

\[
(7)
\]

which becomes, by simple transformation:

\[
(8)
\]

where \( \alpha \) if \( 0 \) and \( \gamma \) is marked by \( n \), so note that this value is determined by block size \( n \), as already mentioned.

If \( p \) is a given a small upper tail probability showing a potential loss, the VaR of a financial position with logarithmic returns is the \( \alpha \) quantile of the sub-period maxima under limiting generalized extreme value distribution:

\[
(9)
\]

where \( n \) is a block size.

2 More detailed in Appendix.

The first distribution is called the Fréchet family, the second is the Weibull family, and the third is the Gumbel family. Let the real constants be \( a_n \) and \( b_n \) (\( a_n \geq 0 \)), so for every \( n \)

\[
\lim_{n \to \infty} P\left( \frac{M_n - b_n}{a_n} \leq x \right) = \lim_{n \to \infty} F^n(a_n x + b_n) = G(x)
\]

applies for non-degenerate distribution function \( G(x) \). When this condition applies, it is said that \( F \) is in the domain of attraction of maxima from \( G \), i.e., \( F \in D(G) \).

The extreme value distribution includes three parameters: \( \gamma \) - shape parameter, \( \beta_n \) - location parameter, and \( \alpha_n > 0 \) - scale parameter. They can be estimated using either parametric or non-parametric methods.

a. Block Maxima Method (BMM)

If we have a data sample of \( T \) returns \( r_i \), the data can be divided into \( g \) blocks, each containing \( n \) data, i.e., \( T = ng \). Let \( r_{n,i} \) be the maximum returns in the \( i \)th block, i.e.,

\[
r_{n,i} = \max_{1 \leq j \leq n} \left\{ r_{(i-1)n+j} \right\}, \quad i = 1, \ldots, g
\]

(6)

where \( 1 \leq j \leq n \), and \( i = 0, \ldots, g - 1 \). \( n \) is typically the number of trading days in one month, quarter, or year.

---

1 Time series of returns.
These block maxima \(\{r_{n,i}\}\) data are used in parameter evaluation of generalized extreme value distribution. It is logical to conclude that this estimation will be influenced by block size \(n\).

Assuming the block maxima fit generalized extreme value distribution, the probability density function of standardized variables \(\frac{r_{n,i} - \beta_n}{\alpha_n}\) is given as (Tsay 2010)

\[
g_\gamma(x) = \begin{cases} 
(1 + \gamma x)^{-1/\gamma-1} \exp\left[-(1 + \gamma x)^{-1/\gamma}\right] & \gamma \neq 0, \\
\exp[-x - \exp(-x)] & \gamma = 0,
\end{cases}
\]  

which becomes, by simple transformation:

\[
g = \begin{cases} 
\frac{1}{\alpha_n}\left[1 + \frac{\gamma_n (r_{n,i} - \beta_n)}{\alpha_n}\right]^{-(1 + \gamma_n)/\gamma_n} \exp\left[-(1 + \frac{\gamma_n (r_{n,i} - \beta_n)}{\alpha_n})^{-1/\gamma_n}\right], & \gamma_n \neq 0, \\
\frac{1}{\alpha_n}\exp\left[-\frac{r_{n,i} - \beta_n}{\alpha_n} - \exp\left(-\frac{r_{n,i} - \beta_n}{\alpha_n}\right)\right], & \gamma_n = 0,
\end{cases}
\]  

where \(1 + \gamma_n (r_{n,i} - \beta_n)/\alpha_n > 0\) if \(\gamma_n \neq 0\). Shape parameter \(\gamma\) is marked by \(n\), so note that this value is determined by block size \(n\), as already mentioned.

If \(p\) is a given a small upper tail probability showing a potential loss, the VaR of a financial position with logarithmic returns \(r_i\) is the \((1 - p)^{th}\) quantile of the sub-period maxima under limiting generalized extreme value distribution\(^2\):

\[
\text{VaR} = \begin{cases} 
\beta_n - \frac{\alpha_n}{\gamma_n} \left\{1 - \left[-n \ln(1 - p)\right]^{-\gamma_n}\right\}, & \gamma_n \neq 0, \\
\beta_n - \alpha_n \ln\left[-n \ln(1 - p)\right], & \gamma_n = 0.
\end{cases}
\]  

where \(n\) is a block size.

\(^2\) More detailed in Appendix.
However, with a larger \( n \), this method better approximates block maxima distribution than generalized extreme value distribution. It also achieves a low bias in parameter estimates. But on the other hand a larger \( n \) means a lower \( g \), for fixed sample size \( T \) we use. And with more blocks - larger \( g \) - the variance obtained for parameter estimates is smaller. Hence, a compromise between these two values is needed.

b. Peaks-over-threshold method (POT)

The traditional approach – the block maxima method – largely dissipates data, because only a single extreme value from every block is used. This is the biggest disadvantage of this model, so in practice it is increasingly being replaced with the method based on peaks-over-threshold, where all data representing extremes are used, in the context of exceeding some high level.

If we mark a certain threshold as \( u \), and we observe the series of daily log returns \( r_t \), then if the \( i \)-th excess happens on the \( i \)-th day, this model focuses on the data ( \( t_i, r_i - u \) ). The basic theory of this new approach observes the conditional distribution of \( r = x + u \), which is for \( r \leq x + u \), given that the threshold is exceeded, \( r > u \), as follows:

\[
P(r \leq x + u | r > u) = \frac{P(u \leq r \leq x + u)}{P(r > u)} = \frac{P(r \leq x + u) - P(r \leq u)}{1 - P(r \leq u)}.
\]

The main distribution used for the modelling of excess over the threshold is the generalized Pareto distribution, defined in the following way:

\[
G_{\gamma, \psi(u)}(x) = \begin{cases} 
1 - \left(1 + \frac{\gamma x}{\psi(u)} \right)^{-1/\gamma}, & \text{if } \gamma \neq 0, \\
1 - \exp\left(-\frac{x}{\psi(u)}\right), & \text{if } \gamma = 0,
\end{cases}
\]

where \( \psi(u) > 0, x \geq 0 \) for \( \gamma \geq 0 \), and \( 0 \leq x \leq -\psi(u)/\gamma \) when \( \gamma < 0 \). Therefore, we conclude that the conditional distribution from \( r \), if \( r > u \), approximates the generalized Pareto distribution well with parameters \( \gamma \) and
\[ \psi(u) = \alpha + \gamma(u - \beta). \] Parameter \( \psi(u) \) is a scale parameter, and \( \gamma \) is a shape parameter.

Generalized Pareto distribution has a very significant feature. If the excess distribution of \( r \) with the given threshold \( u_0 \) is a generalized Pareto distribution with shape parameter \( \gamma \) and scale parameter \( \psi(u_0) \), then for arbitrary threshold \( u > u_0 \) the given excess distribution for threshold \( u \) is also a generalized Pareto distribution, with shape parameter \( \gamma \) and scale parameter \( \psi(u) = \psi(u_0) + \gamma(u - u_0) \) (Mladenovic and Mladenovic 2006).

When the parameter \( \gamma = 0 \), then the generalized Pareto distribution is an exponential distribution. Therefore, it is suggested that a graphic examination of the tail behaviour using a QQ plot be carried out. If \( \gamma = 0 \), then the graph of the excess is linear.

The peaks-over-threshold model has a problem regarding the choice of an adequate threshold. In practice the given problem is usually solved in the following way.

For a given high threshold \( u_0 \), let the excess \( r - u_0 \) follow generalized Pareto distribution with parameters \( \gamma \) and \( \psi(u_0) \), where \( 0 < \gamma < 1 \). Then the mean excess over threshold \( u_0 \) is:

\[ E(r - u_0 | r > u_0) = \frac{\psi(u_0)}{1 - \gamma}. \] (12)

The mean excess function, \( e(u) \), is defined, for every \( u > u_0 \), as:

\[ e(u) = E(r - u | r > u) = \frac{\psi(u_0) + \gamma(u - u_0)}{1 - \gamma}. \] (13)
Therefore, for given value $\gamma$, the mean excess function is a linear function of excess $u - u_0$. Hence, a simple graphic model is used for the determination of given threshold $u_0$, forming the empirical mean excess function as

$$e_T(u) = \frac{1}{N_u} \sum_{i=1}^{N_u} (r_{i,t} - u),$$

(14)

where $N_u$ is the number of returns exceeding the threshold $u$, and $r_{i,t}$ are the values of the given returns. Threshold $u$ is chosen so that the empirical mean excess function is approximately linear for $r > u$.

For the given probability $p$ in the upper tail, the $(1-p)$-quantile of log return $r_t$ is

$$\text{VaR} = \begin{cases} \beta - \frac{\alpha}{\gamma} \left[ 1 - \left( -D \ln(1 - p) \right)^\gamma \right] & \gamma \neq 0, \\ \beta - \alpha \ln \left( -D \ln(1 - p) \right) & \gamma = 0, \end{cases}$$

(15)

where $D$ is usually the number of trading days in the baseline time interval (252 for a year, etc.). The VaR estimate is much more stable when using the POT because with the traditional BMM approach the VaR is very sensitive to changes in the size of blocks $n$.

c. Backtesting

The procedure for evaluating risk measures and the accuracy of the models used is called backtesting. This process of estimating if the amount of losses predicted by VaR is correct implements unconditional or conditional coverage tests for the correct number of exceedances. The unconditional tests control if the frequency exceptions, during the selected time interval, are in accordance with the chosen confidence level. The most commonly used test is the Kupiec test. Meanwhile, conditional coverage tests examine conditionality and changes in data over time (Jorion 2007), and the most famous test in this group is the Christoffersen independence test.
(a) Kupiec test

Let $N$ be the observed number of exceedances in the sample, or in other words the number of days over a $T$ period of time when the portfolio loss was larger than the VaR estimate, and the observed excess rate is $\hat{p} = \frac{N}{T}$. The ratio of failures, $N$, to the trials, $T$, under the null hypothesis should be $p$, and the failure number follows a binomial distribution.

The basic idea is to determine if the observed excess rate is significantly different from $p$, the excess rate determined by the given confidence level. According to Kupiec (1995), the proportion-of-failures test (POF) is best implemented as a likelihood ratio test. The appropriate likelihood ratio statistic is

$$LR_{uc} = 2 \ln\left[(1 - \frac{N}{T})^{T-N} \left(\frac{N}{T}\right)^N\right] - 2 \ln\left[(1 - p)^{T-N} p^N\right].$$

The Kupiec test has a chi-square distribution, asymptotically, with one degree of freedom. This test can reject a model for both high and low failures but, as stated by Kupiec (Kupiec 1995), its power is generally poor, so conditional coverage tests such as the Christoffersen test can be used for further examining VaR model reliability.

(b) Christoffersen independence test

The Christoffersen independence test is important insofar a sit detects whether or not the exceptions occur in clusters. In other words, it can observe if the probability of exceptions on any day depends on the outcome of the previous day. If the existence of clustering can be proved, the model is misspecified and needs to be recalibrated. Let $n_{ij}$ be the number of days when outcome $j$ occurs after outcome $i$ occurred the day before. The probability of state $j$ being observed given that state $i$ was observed the previous day is noted by $\pi_{ij}$ (Jorion 2007). The test statistic testing independence is (Dowd 2005):

$$LR_{ind} = -2 \ln\left[(1 - \pi)^{n_{00} + n_{01}} \pi^{n_{01} + n_{11}}\right] + 2 \ln\left[(1 - \pi_{01})^{n_{00} - n_{01}} \pi_{01}^{n_{01} - n_{11}} (1 - \pi_{11})^{n_{01} - n_{11}} \pi_{11}^{n_{11} - n_{01}}\right].$$

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where the corresponding probabilities are $\pi_{ij} = \frac{n_{ij}}{\sum_{j} n_{ij}}$, so $\pi_{01}$ is the probability of a non-exception being followed by an exception, and $\pi_{11}$ is the probability of an exception being followed by an exception. The absolute probability of a non-exception or exception being followed by an exception is denoted by $\pi$. The test statistic is distributed as $\chi^2$ with two degrees of freedom.

The Christoffersen test needs several hundred observations in order to be accurate, and the main advantage of this procedure is that it can reject a VaR model that generates either too many or too few clustered violations. These two LR tests can be combined, thereby creating a complete test for coverage and independence which is also distributed as a $\chi^2(2)$:

$$LR_{cc} = LR_{uc} + LR_{ind}.$$  \hspace{1cm} (18)

This is the Christoffersen approach to check the predictive ability and accuracy of a VaR model. The advantage of this test is in the combination of two tests that can be tested separately to backtrack if the model fails due to wrong coverage or due to exception clustering. Specifically, it is possible for the model to pass the joint test while still failing each of the individual tests (Campbell 2005). Therefore these tests separately provide the necessary tools to evaluate and compare the VaR models mentioned above.

To illustrate these methods, we will use data that are explained in the next section.

**4. DATA AND DESCRIPTIVE STATISTICS**

We examined the daily log returns of the Montenegrin stock index MONEX20, which is the best indicator of the state of the Montenegrin stock market. The stock market index MONEX20 consists of 20 stock market and free market issuers. Specific criteria have established that MONEX20 gets the most liquid stocks in order to best reflect price movements in the Montenegrin market. MONEX20 is a weighted index – each issuer's share is determined by its capitalization. The market capitalization includes ordinary shares that are free
floating. Shares of the twenty highest-ranked companies by liquidity ratio constitute the MONEX20. The MONEX20 index is calculated according to the following formula

\[
MONEX20 = \frac{\sum_{i=1}^{n} p_{i,t} \cdot q_{i,R}}{\sum_{i=1}^{n} p_{i,0} \cdot q_{i,R}} \cdot 1000 \cdot C_T, \tag{19}
\]

where \( p_{i,t} \) is the price of the \( i \)th stock on day \( t \), \( p_{i,0} \) is the base price of the \( i \)th stock on the date of formation of the index, \( q_{i,R} \) is the number of free-floating stocks, and \( C_T \) is the correction factor for ensuring continuity of the index in the time before calculation of the index according to the new composition.

VaR-estimated values based on two different methods are compared. The time series of observed log returns of the MONEX20 stock index on a daily basis consists of 2,508 data in total (from 5th January 2004 to 21st February 2014). Log daily returns (or continuously compounded returns) represent the difference between the logarithmic level of prices on two successive days. It can also be expressed in percentages, when these differences are multiplied by 100. The data are taken from the website of the Montenegro Stock Exchange (http://www.montenegroberza.com). Empirical results are obtained by using program package R.

The expressed volatility and fat-tail nature of the Montenegrin stock index MONEX20, which comprises the 20 most liquid stocks in the Montenegrin capital market, can be seen in Graph 1. Its empirical distribution deviates from normal distribution, as the Q-Q plot (Graph 2) shows, as well as from the coefficient of skewness and Jarque-Bera test-statistics (JB). These descriptive statistics are given in Table 1 with corresponding \( p \)-values in parenthesis.

**Table 1.** Basic descriptive statistics of daily logarithmic return for MONEX20

<table>
<thead>
<tr>
<th>Variance</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>JB ( (p &lt; 10^{-4}) )</th>
<th>Box-Ljung ( (m=10) )</th>
<th>Box-Ljung ( \chi^2_{10} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.869291</td>
<td>0.6863</td>
<td>6.5368</td>
<td>4672.7</td>
<td>219.636</td>
<td>1003.793</td>
</tr>
</tbody>
</table>

**Source:** Authors’ calculation

**Note:** \( p \)-values are in parentheses.
5. EMPIRICAL RESULTS

This part of the paper presents the results of the empirical research that focuses on the application of the extreme value theory to the emerging market of Montenegro. Extreme value theory has been extended to serially dependent observations, provided that the dependence is weak (Tsay 2010). The same form of limiting extreme value distribution holds for stationary normal sequences, provided that the autocorrelation function of observations is square-summable (Berman 1964). However, bearing in mind that distribution tail modelling can result in autocorrelation and heteroscedasticity sensitivity in data, we overcame this problem by pre-filtering the log returns. We estimated the AR(2)-GARCH(1,1) model for the series of original log returns of the MONEX20 stock index, and then applied two EVT models from the ARMA-GARCH family to the residuals of this model. This is a proposed two-stage method consisting of modelling the conditional distribution of asset returns against the current volatility and then fitting the generalised extreme value distribution on the tails of residuals that are independent and identically distributed. (McNeil and Frey 2000).

Firstly, VaR-estimated values based on the block maxima method are shown. The Hill estimator is used for the shape parameter. Graph 3 shows the scatterplots of the Hill estimator with its 95% confidence interval. It is obvious that the estimator is stable for a bigger $q$ for both positive and negative residuals. The scatterplots also show that the shape parameter is larger for the negative residuals, which indicates that the series has a heavier left tail. Therefore the distribution of residuals belongs to the Fréchet family of distribution.

Graph 1. Daily return of MONEX20 stock index

Source: Montenegro stock exchange and Authors’ calculation, February 2014

Graph 2. Q-Qplot of daily return of MONEX20 relative to normal distribution

Source: Authors’ calculation

The skewness shows that the series is not particularly asymmetric, but normality deviation is due to high kurtosis, which means ‘fat tails’ exist – the tails are heavier than normal distribution tails.
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Table 2. Maximum-likelihood estimates of extreme value distribution parameters for AR(2)-GARCH(1,1) model residuals

<table>
<thead>
<tr>
<th>Sub-period length</th>
<th>Minimal returns</th>
<th>Maximal returns</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha_n$</td>
<td>$\beta_n$</td>
</tr>
<tr>
<td>1 month ($n=21$)</td>
<td>1.007 (0.098)</td>
<td>1.461 (0.208)</td>
</tr>
<tr>
<td>1 quarter ($n=63$)</td>
<td>1.747 (0.112)</td>
<td>2.973 (0.269)</td>
</tr>
<tr>
<td>6 months ($n=126$)</td>
<td>0.359 (0.109)</td>
<td>0.109 (0.151)</td>
</tr>
</tbody>
</table>

Source: Authors’ calculation

The maximum-likelihood method is applied in parameter estimation of generalized extreme value distribution. Table 2 summarizes the estimation results for different choices of sub-period length, from 1 month ($n=21$) to half a year ($n=126$). Based on the obtained results, it can be concluded that estimates of $\alpha_n$ and $\beta_n$ increase as sub-period length $n$ increases, as expected, and opposed to the shape parameter estimates that decrease. Ultimately, the standard errors of parameter estimates become significantly higher as $n$ increases.

Using the probability density function of generalized extreme value distribution, it can be shown that the residuals of generalized extreme value distribution should form a random sample of exponentially distributed random variables so the fitted model is correctly specified. Graph 4 shows the residuals of generalized extreme value distribution fitted to AR(2)-GARCH(1,1) model residuals for a sub-period length of 21 days. The diagram above gives the residuals and the diagram below shows a Q-Q plot against exponential distribution, so the fit is reasonable, based on this plot.

Source: Authors’ calculation
Table 2. Maximum-likelihood estimates of extreme value distribution parameters for AR(2)-GARCH(1,1) model residuals

<table>
<thead>
<tr>
<th>Sub-period length</th>
<th>$\alpha_n$</th>
<th>$\beta_n$</th>
<th>$\gamma_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Minimal returns</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>month ($n=21, g=120$)</td>
<td>1.007 (0.098)</td>
<td>1.747 (0.112)</td>
<td>0.359 (0.109)</td>
</tr>
<tr>
<td>quarter ($n=63, g=40$)</td>
<td>1.461 (0.208)</td>
<td>2.973 (0.269)</td>
<td>0.109 (0.151)</td>
</tr>
<tr>
<td>6 months ($n=126, g=20$)</td>
<td>1.643 (0.388)</td>
<td>3.671 (0.482)</td>
<td>0.036 (0.324)</td>
</tr>
<tr>
<td><strong>Maximal returns</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>month ($n=21, g=120$)</td>
<td>1.114 (0.105)</td>
<td>1.947 (0.119)</td>
<td>0.364 (0.093)</td>
</tr>
<tr>
<td>quarter ($n=63, g=40$)</td>
<td>1.608 (0.268)</td>
<td>3.306 (0.316)</td>
<td>0.281 (0.199)</td>
</tr>
<tr>
<td>6 months ($n=126, g=20$)</td>
<td>2.333 (0.573)</td>
<td>4.283 (0.705)</td>
<td>0.017 (0.349)</td>
</tr>
</tbody>
</table>

Source: Authors’ calculation

Graph 4. Residual plot from fitting generalized extreme value distribution to negative AR(2)-GARCH(1,1) model residuals, for a sub-period length of 21 days

Source: Authors’ calculation

Now we can calculate VaR following formula (9). If the probability is 0.01, the corresponding VaR for negative log returns and $n=63$, based on the results from Table 3, is
Table 4 contains the evaluated parameters, α and γ, for the given data sets, with a given threshold variation between 2% and 3%. Given parameters are used for the calculation of VaR and the adequacy of the model can be seen in the plots in Graphs 6 and 7.

Graphs 6 and 7 show highly adjusted generalized Pareto distribution to negative residuals. Graph 6 shows exceeding fit to generalized Pareto distribution (upper plot), as well as the tail probability estimate (lower plot). In Graph 7 we present the Q-Q plot (lower plot), which contains empirical quantiles that form a straight line, while the upper plot is a scatter plot of residuals. Hence, the empirical quantiles form an approximately straight line, and we have one more indicator leading to the conclusion that negative residuals are properly modelled by generalized Pareto distribution.

If we choose a sub-period length of 21 days, the VaR for the negative log returns and the same probability is then

\[ \text{VaR} = 2.973 - \frac{1.461}{0.109} \left[ 1 - \left[ -63 \ln(1 - 0.01) \right]^{-0.109} \right] = 3.6576. \]

Therefore, based on the same formula, VaR can be calculated for the probability 0.05, which will be used to compare the results with the other method. Given the sub-period length of 21 days and the probability 0.05, the VaR for negative residuals is 1.6731. These results are given in percentages, and for the sub-period length of 21 days we give estimates of the VaR based on the block maxima method in Table 3. We compare the results obtained using AR(2)-GARCH(1,1) model residuals and results using the MONEX20 stock index’s original series of log returns, and the results are almost identical.

Table 3. Evaluation of VaR based on Block maxima method

<table>
<thead>
<tr>
<th>Probability</th>
<th>VaR for AR(2)-GARCH(1,1) residuals</th>
<th>VaR for original series of log returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
<td>0.0167</td>
<td>0.0161</td>
</tr>
<tr>
<td>0.99</td>
<td>0.0385</td>
<td>0.0380</td>
</tr>
</tbody>
</table>

Source: Authors’ calculation

We then evaluated VaR using the peaks-over-threshold method. The negative AR(2)-GARCH(1,1) model residuals obtained are observed, and selection for threshold \( u \) is based on the graph of mean excess function. The Q-Q plot included in Graph 5 shows that coefficient \( \gamma \neq 0 \). Also, the mean excess function graph has a linear tendency from the threshold level of 2%-3%, so we give results for 3 varying threshold values, 2%, 2.5%, and 3%.

The set of extreme events exceeding the 2.5% threshold has 114 data. For thresholds of 2% and 3% the numbers of exceeding events are 163 and 76, respectively. The extreme value distribution is modelled based on these data.

\[ \text{VaR} = 1.747 - \frac{1.007}{0.359} \left[ 1 - \left[ -21 \ln(1 - 0.01) \right]^{-0.359} \right] = 3.8451. \]
sets. Table 4 contains the evaluated parameters $\gamma, \alpha$, and $\beta$ for the given data sets, with a given threshold variation between 2% and 3%. Given parameters are used for the calculation of VaR and the adequacy of the model can be seen in the plots in Graphs 6 and 7.

Graphs 6 and 7 show highly adjusted generalized Pareto distribution to negative residuals. Graph 6 shows exceeding fit to generalized Pareto distribution (upper plot), as well as the tail probability estimate (lower plot). In Graph 7 we present the Q-Q plot (lower plot), which contains empirical quantiles that form a straight line, while the upper plot is a scatter plot of residuals. Hence, the empirical quantiles form an approximately straight line, and we have one more indicator leading to the conclusion that negative residuals are properly modelled by generalized Pareto distribution.

**Graph 5.** Q-Q plot of excess residuals over a 2.5% threshold and mean excess plot for negative AR(2)-GARCH(1,1) model residuals

![Graph 5](image)

**Source:** Authors’ calculation
The peaks-over-threshold-method results for the VaR are summed up in Table 5. It is concluded that the estimated parameter results are more stable than the block maxima method, which shows parameter assessment variations depending on the choice of block size $n$. It is evident that the results of VaR differ less depending on the different values of threshold excess, and within the same confidence level. We compared the obtained results using AR(2)-GARCH(1,1) model residuals and results using the original series of log returns of the MONEX20 stock index, and again these results are almost identical.

**Graph 6.** Plots for generalized Pareto distribution to residuals

**Table 4.** Result estimates of two-dimensional Poisson process of AR(2)-GARCH(1,1) residuals

<table>
<thead>
<tr>
<th>Threshold</th>
<th>Number of exceeding</th>
<th>$\gamma_n$</th>
<th>$\alpha_n$</th>
<th>$\beta_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3%</td>
<td>76</td>
<td>-0.097 (0.084)</td>
<td>0.0227 (0.008)</td>
<td>-0.037 (0.016)</td>
</tr>
<tr>
<td>2.5%</td>
<td>114</td>
<td>0.014 (0.066)</td>
<td>0.0131 (0.003)</td>
<td>-0.016 (0.006)</td>
</tr>
<tr>
<td>2%</td>
<td>163</td>
<td>0.006 (0.059)</td>
<td>0.0135 (0.003)</td>
<td>-0.017 (0.005)</td>
</tr>
</tbody>
</table>

**Source:** Authors’ calculation

**Note:** Standard errors are in parentheses.
**Graph 7.** Plots for generalized Pareto distribution to residuals – scatterplot of residuals and q-q plot of residuals

![Graph 7](image)

**Source:** Authors’ calculation

The Value at Risk estimates used to compare the results are as follows: if we possess €1000-worth of stocks described by the stock market index MONEX20, with probability 0.05, meaning that there is a 95% probability the loss will be lower or the same as the VaR on the following trading day, the parameter estimated value is €16.731 when using the block maxima method (sub-period length is 21), and €23.7 when using the peaks-over-threshold method (threshold is 2.5%). The corresponding VaR with the probability 0.01 is €38.451 using BMM (sub-period length is 21), and €45.9 using POT (threshold is 2.5%).

**Table 5.** Evaluation of VaR based on peaks-over-threshold method

<table>
<thead>
<tr>
<th>Threshold</th>
<th>p-value</th>
<th>VaR for AR(2)-GARCH(1,1) residuals</th>
<th>VaR for original series of log returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5%</td>
<td>0.05</td>
<td>0.0237</td>
<td>0.0238</td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>0.0459</td>
<td>0.0463</td>
</tr>
<tr>
<td></td>
<td>0.001</td>
<td>0.0787</td>
<td>0.0770</td>
</tr>
<tr>
<td>2%</td>
<td>0.05</td>
<td>0.0236</td>
<td>0.0237</td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>0.0460</td>
<td>0.0467</td>
</tr>
<tr>
<td></td>
<td>0.001</td>
<td>0.0785</td>
<td>0.0765</td>
</tr>
</tbody>
</table>
In the second part of Table 6 we determine whether or not the estimated VaR based on the block maxima method suffers from volatility clustering by using the Christoffersen tests. The Christoffersen independence test and the joint test give the same results: estimates of VaR based on BMM reject the null hypothesis of no exceedance dependence and no volatility clustering respectively, and are therefore disqualified, with confidence levels of 95% and 99%. Since we already know that the Kupiec test produced results where critical values were exceeded significantly, the results from the joint test are not surprising. It follows from the rejection of $LR_{uc}$ as well as $LR_{ind}$ that a combined hypothesis of correct conditional coverage can be safely rejected.

<table>
<thead>
<tr>
<th>Type of test</th>
<th>95%</th>
<th>99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>the Kupiec test</td>
<td>expected number of exceedances</td>
<td>125</td>
</tr>
<tr>
<td></td>
<td>actual number of exceedances</td>
<td>205</td>
</tr>
<tr>
<td></td>
<td>test statistic</td>
<td>45.005 (&lt;=10^{-4})</td>
</tr>
<tr>
<td>the Christoffersen independence test</td>
<td>test statistic</td>
<td>13.531 (&lt;=10^{-3})</td>
</tr>
<tr>
<td>the Christoffersen joint test</td>
<td>test statistic</td>
<td>113.336 (0)</td>
</tr>
</tbody>
</table>

**Source:** Authors’ calculation

The accuracy of the estimates of the models considered in the present study can be assessed by counting the number of actual returns that are larger than the estimated VaR, and comparing this figure with the theoretically expected number of excesses for a determined probability. Of course, the closer the empirically observed number of excesses is to the theoretically expected amount, the more preferable the method is for estimating risk measures.

**Table 6. Backtesting of VaR estimates by Block Maxima method**

The accuracy of the estimates of the models considered in the present study can be assessed by counting the number of actual returns that are larger than the estimated VaR, and comparing this figure with the theoretically expected number of excesses for a determined probability. Of course, the closer the empirically observed number of excesses is to the theoretically expected amount, the more preferable the method is for estimating risk measures.

In the first part of Table 6 the results of the Kupiec test are presented with the number of excesses for different quantiles associated with distribution under the block maxima method, together with the theoretically expected number of excesses for the MONEX20 index. As can be seen from the table, the block maxima method fails to accurately measure Value at Risk with a probability of either 0.01 or 0.05, and the actual number of excesses far surpasses the theoretically expected number.
In the second part of Table 6 we determine whether or not the estimated VaR based on the block maxima method suffers from volatility clustering by using the Christoffersen tests. The Christoffersen independence test and the joint test give the same results: estimates of VaR based on BMM reject the null hypothesis of no exceedance dependence and no volatility clustering respectively, and are therefore disqualified, with confidence levels of 95% and 99%. Since we already know that the Kupiec test produced results where critical values were exceeded significantly, the results from the joint test are not surprising. It follows from the rejection of $LRuc$ as well as $LRind$ that a combined hypothesis of correct conditional coverage can be safely rejected.

Table 7. Backtesting of VaR estimates by peaks-over-threshold method

<table>
<thead>
<tr>
<th>Type of test</th>
<th>VaR (2%)</th>
<th>VaR (2.5%)</th>
<th>VaR (3%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>the Kupiec test 95%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>expected.number of exceedances</td>
<td>125</td>
<td>125</td>
<td>125</td>
</tr>
<tr>
<td>actual number of exceedances</td>
<td>128</td>
<td>127</td>
<td>146</td>
</tr>
<tr>
<td>test statistic</td>
<td>0.056</td>
<td>0.021</td>
<td>3.391</td>
</tr>
<tr>
<td></td>
<td>(0.812)</td>
<td>(0.884)</td>
<td>(0.065)</td>
</tr>
<tr>
<td>the Kupiec test 99%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>expected.number of exceedances</td>
<td>25</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>actual number of exceedances</td>
<td>28</td>
<td>28</td>
<td>26</td>
</tr>
<tr>
<td>test statistic</td>
<td>0.331</td>
<td>0.331</td>
<td>0.034</td>
</tr>
<tr>
<td></td>
<td>(0.565)</td>
<td>(0.565)</td>
<td>(0.854)</td>
</tr>
<tr>
<td>the Christoffersen test 95%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>test statistic for independence test</td>
<td>19.755</td>
<td>19.763</td>
<td>24.544</td>
</tr>
<tr>
<td></td>
<td>(&lt;10^-4)</td>
<td>(&lt;10^-4)</td>
<td>(&lt;10^-4)</td>
</tr>
<tr>
<td>test statistic for joint test</td>
<td>46.887</td>
<td>47.673</td>
<td>49.999</td>
</tr>
<tr>
<td></td>
<td>(&lt;10^-4)</td>
<td>(&lt;10^-4)</td>
<td>(&lt;10^-4)</td>
</tr>
<tr>
<td>the Christoffersen test 99%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>test statistic for independence test</td>
<td>32.018</td>
<td>32.018</td>
<td>29.736</td>
</tr>
<tr>
<td></td>
<td>(&lt;10^-4)</td>
<td>(&lt;10^-4)</td>
<td>(&lt;10^-4)</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
<td>(0.001)</td>
<td>(0.008)</td>
</tr>
</tbody>
</table>

Source: Authors’ calculation

Note: p-values are in parentheses.
The results for the POT model are significantly better. Table 7 shows that VaR violations based on the POT model are far better than in the case of BMM. Unlike BMM, POT models passed the Kupiec test with 95% and 99% confidence levels, for all selected threshold values (2%, 2.5%, and 3%), so this model gives far better VaR estimates regarding unconditional coverage (actual number of exceedances are close to the expected number). On the other hand, the POT model did not pass the Christoffersen test at either the 99% or 95% confidence level. The Cristoffersen independence test showed that estimates of VaR based on the POT model reject the null hypothesis of independent exceedances for both 95% and 99%. Finally, this model also fails to pass the Christoffersen joint test for both 95% and 99%. From these results we conclude that the POT model for measuring VaR is not appropriate for capturing volatility clustering in the Montenegrin stock market and should be improved, though it is accurate as far as correct exceedances are concerned.

The choice of tail distribution probability also has an important role in the calculation of VaR. It is evident that with the increase of the confidence level from 95% to 99% the block maxima method estimates are significantly improved, but cannot outperform the peaks-over-threshold method.

Also, in the case of the latter approach (Table 5), it is noticeable that less reliable VaR estimates are obtained for a very low 0.1% probability. Therefore, that significance level was not discussed in the POT method.

6. CONCLUSION

Investors and risk managers have become more concerned with events occurring under extreme market conditions. In this study, EVT methods are used to model tail returns and estimate the Value at Risk of the MONEX20 stock index over the period from 5th January 2004 to 21st February 2014. We hope to provide international investors with another risk perspective. The financial turmoil in the Montenegrin stock market at the end of 2007 and in 2008 indicates the need for investors and financial institutions to understand and model the extreme return distributions of the financial market and to generate VaR estimates in order to manage and control extreme risk. This paper offers investors a number of interesting findings. The performance of the POT model in measuring the VaR of the Montenegrin stock index MONEX20 is still
much better than the VaR estimates obtained by BMM, particularly in terms of the number of exceedances of VaR estimates, since estimates of VaR based on the POT model passed the Kupiec test. Furthermore, we showed that both the BMM and the POT model fail to pass the Christoffersen independence and conditional coverage tests for accuracy in Value at Risk measuring. It is obvious that the block maxima method underestimates the VaR, since it did not pass any of the three tests considered. Only hit sequences (formed by exceedances) that satisfy both unconditional coverage and independence properties can be described as evidence of an accurate VaR model (Campbell, 2005). So, although better, the POT model still cannot be treated as an accurate VaR model for the Montenegrin stock market.

However, the turbulent market of 2008 inevitably causes problems in estimating parameters to describe future market movements. Abnormal market behaviour is simply beyond what any VaR model is intended to capture. After the crisis, many stock markets’ VaR values became riskier, and, accordingly, investors and fund managers should be aware of such changing market dynamics when formulating portfolio strategies that include the Montenegrin stock market. An important contribution of this study is the significant evidence supporting extreme value theory and its performance in calculating VaR in the Montenegrin frontier stock market. Bearing in mind that Montenegrin stock market returns exhibit extreme behaviour can help investors to understand the distribution of market returns better and obtain more accurate return forecasts. In this respect our results also have practical implications, because they suggest risk management should include extreme risks. In further analysis it would be interesting to include more stock exchange indices from similar markets in South-Eastern and Central Europe in order to see whether this conclusion can be generalized to other frontier and emerging markets.
APPENDIX

VaR estimation using the block maxima method

The maximum-likelihood method assumes the block size is large enough for observations of block maxima to be independent, whether or not the original data are independent (McNeil, et al, 2005). Then the likelihood function, by logarithmic transformation, is:

\[
\ell(r_{n,1}, \ldots, r_{n,g} \mid \gamma_n, \alpha_n, \beta_n) = \prod_{i=1}^{g} f(r_{n,i})
\]

\[
= \sum_{i=1}^{g} \ln f(r_{n,i})
\]

(A1)

\[
= -g \ln \alpha_n - \left(1 + \frac{1}{\gamma'} \right) \sum_{i=1}^{g} \ln \left(1 + \gamma \frac{r_{n,i} - \beta_n}{\alpha_n} \right) - \sum_{i=1}^{g} \left(1 + \gamma \frac{r_{n,i} - \beta_n}{\alpha_n} \right)^{-1/\gamma'.}
\]

These estimates are consistent and asymptotically efficient (as Smith has shown in 1985 in the case of \(\gamma > -1/2\)).

Let \(p^*\) be a small upper tail probability showing a potential loss, and \(r_{n}^*\) be the \((1-p^*)^{th}\) quantile of the sub-period maxima under the limiting generalized extreme value distribution. Then,

\[
1 - p^* = \begin{cases} 
\exp\left\{- \left[1 + \frac{\gamma_n (r_{n}^* - \beta_n)}{\alpha_n} \right]^{-1/\gamma_n} \right\}, & \gamma_n \neq 0, \\
\exp\left[-\exp\left(-\frac{r_{n}^* - \beta_n}{\alpha_n}\right)\right], & \gamma_n = 0,
\end{cases}
\]

(A2)

where \(1 + \gamma_n (r_{n}^* - \beta_n) / \alpha_n > 0\) if \(\gamma_n \neq 0\). By logarithmic transformation, we have:
The quantile $r^*_n$ for a given probability $p^*$ is the VaR for the sub-period maximum. Knowing that most asset returns have either weak serial correlations or no correlations at all, the relation between sub-period maxima and the observed return series $r_i$ is as follows

$$1 - p^* = P(r_{n,i} \leq r^*_n) = \left[P(r_i \leq r^*_n)^n\right]^p. \quad (A5)$$

and the quantile is

$$r^*_n = \begin{cases} 
\beta_n - \frac{\alpha_n}{\gamma_n} \left\{1 - \left[-\ln(1 - p^*)\right]^{\gamma_n}\right\} & \gamma_n \neq 0, \\
\beta_n - \alpha_n \ln\left[-\ln(1 - p^*)\right] & \gamma_n = 0.
\end{cases} \quad (A4)$$

REFERENCES


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