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IMPOSSIBILITY THEOREMS IN MULTIPLE VON WRIGHT'S PREFERENCE LOGIC**

ABSTRACT: *By using the standard combining logics technique (D. M. Gabbay 1999) we define a generalization of von Wright's preference logic (G. H. von Wright 1963) enabling to express, on an almost propositional level, the individual and the social preference relations simultaneously. In this context we present and prove the counterparts of crucial results of the Arrow-Sen social choice theory, including impossibility theorems (K. Arrow 1951 and A. K. Sen 1970b), as well as some logical interdependencies between the dictatorship condition and the Pareto rule, and thus demonstrate the power and applicability of combining logics method in mathematical economics.*

KEY WORDS: *impossibility theorems; social choice; preference logic; combining logics.*

JEL CLASSIFICATION: D70, D71.

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1. Introduction

In recent literature D. M. Gabbay was mainly responsible for introducing, clearly defining, and developing the combining logics concept (D. M. Gabbay 1995, 1996a, 1996b and 1999 and D. M. Gabbay, M. Finger 1996), and many subsequent authors then used and applied it. This fruitful concept, particularly when augmented by the fibring semantic approach (D. M. Gabbay 1995, 1996a, 1996b and 1999), provides an application of logical formalism in a widespread scientific discipline spectrum. In this case we demonstrate how the combining logics technique works in the social choice theory context, making it possible to present and interpret some well known results in a more approachable and formally pure way.

From the point of view of formal logic, the original Arrow–Sen theory (K. Arrow 1951 and A. K. Sen 1970b), motivated by social choice procedures, belongs to higher–order theories — even though it is deeply mathematically founded and argued — as individuals and alternatives share the same level in the quantified formulae (see also R. Routley 1979).

On the other hand, based on ideas circulated in mathematical economics and social choice theory, von Wright was the first author to study the logic of preference as a pure logical concept (G. H. von Wright 1963 and 1972) based on a slightly extended propositional language.

The aim of this paper is to present the crucial results of the Arrow–Sen theory, impossibility theorems, in the context of combined von Wright's preference logic, by interpreting the basic Arrow and Sen axioms as almost pure propositional formulae. This approach, although very formal, provides a simple modification and presentation of impossibility results. In order to attain this goal we introduce a generalized version of von Wright's preference logic, obtained by standard combining techniques (D. M. Gabbay 1999), enabling us to express simultaneously individual and social preference relations on the same level.

The paper is organized as follows. In the first part we present the original formulations of axioms as introduced by K. Arrow (K. Arrow 1951) and A. Sen (A. K. Sen 1970) and their slight transformations. We then present the basic elements of G. H. von Wright's original logic of preference (G. H. von Wright 1963 and 1972). In the sequel we develop the formal logical framework precisely, including a combining von Wright's systems enabling us to express finitely many preference relations simultaneously within one logical system. Due to the fact that combining technique is applied to Hilbet–type formulations of slight extensions of the classical propositional system, it is not difficult to justify this process. This framework of combined preference logics will be sufficient to demonstrate, in a simple and approachable way, that, for instance, as indicated in Boričić (2012), some modified forms of impossibility theorems can be obtained. Namely, we isolate the minimal sets of hypotheses, based on von Wright's preference logic, under which it is possible to perceive the spirit of Arrow's and Sen's glorious theorems. In order to make possible a pure formal treatment of axioms, we 'translate' them into the language of von Wright's logic. This operation, for instance, similarly as in B. Boričić (2007) and (2009), results in the replacement of *one* of Arrow's original non–dictatorship axiom **ND** by a *finite number* (number of all individuals) of the corresponding non–dictatorship axioms over the language of von Wright's logic. *One* of Sen's original liberalism axiom **L** will also be replaced by a *finite number* of the corresponding almost propositional form of liberalism axioms.

2. A Description of Arrow–Sen Social Choice Theory

Let V be a set consisting of n individuals. The individuals from V will be denoted by $i, j, ...$ The set of possible alternatives $x, y, z, ...$ is denoted by X. The weak preference relation, a linear and transitive binary relation, expressing preferences of the person i will be R_i and the corresponding social weak preference relation, generated by $(R_1, \dots, R_n)$, will be denoted by R. By P_i and P we denote the strict preference relations (asymmetric, linear and transitive binary relations) induced by R_i and R, respectively, as follows: $P_i = R_i \cap$

$\overline{R_i^{-1}}$ and $P = R \cap \overline{R^{-1}}$. The intended meaning of xPy is 'x is preferred to y'. The way of defining the social preference relation R generated by individual preferences $(R_1, \ldots, R_n)$ is usually called *the social welfare function.* The central problem of Arrow's theory is: does the social welfare function exist, under the conditions described below?

By **IIA** we denote 'the independence of irrelevant alternatives': for each $Y \subseteq X$,

$$(\forall x, y \in Y)(\forall i \in V)(xR_iy \leftrightarrow xR'_iy) \rightarrow \\ \rightarrow (\forall x \in Y)((\forall y \in Y)xRy \leftrightarrow (\forall y \in Y)xR'y)$$

or, since $C(Y, R) = \{x | x \in Y \wedge (\forall y \in Y)xRy\}$, we have

$$(\forall x, y \in Y)(\forall i \in V)(xR_iy \leftrightarrow xR'_iy) \rightarrow C(Y, R) = C(Y, R')$$

Also, a general condition **U** of 'unrestricted domain', requiring that the procedure of generating a social preference P can be applied to any configuration of rational individual preferences P_i, is supposed to hold.

Two additional basic conditions of Arrow's theory 'non–dictatorship' **ND** and the Pareto rule **P**, respectively, are usually presented as follows (K. Arrow 1951, R. Routley 1979, and A. K. Sen 1970b and 1995):

ND $\quad \neg(\exists i \in V)(\forall x, y \in X)(xP_iy \rightarrow xPy)$

P $\quad (\forall x, y \in X)((\forall i \in V)xP_iy \rightarrow xPy)$

The non–dictatorship axiom states that there is no person i, dictator, having such power that, for each two alternatives x and y, if i prefers x to y, the society must prefer x to y as well. The Pareto rule says that, if every individual prefers x to y, then the society must prefer x to y.

Sen has taken into the consideration 'the liberalism axiom' **L** (A. K. Sen 1970a and 1995):

L $\quad (\forall i \in V)(\exists x, y \in X)(x \neq y \wedge (xP_iy \rightarrow xPy) \wedge (yP_ix \rightarrow yPx))$

Note that if we omit the condition $x \neq y$ in the liberalism axiom, then, for $x = y$, this axiom would hold universally, for trivial reasons, because neither condition xP_iy, nor yP_ix would be satisfied. The liberalism condition provides that each individual is decisive over at least one pair of distinct alternatives.

Let us note that the given axioms are not expressed in the first–order language.

The non–dictatorship and Pareto rules were originally introduced by K. Arrow, and the liberalism axiom was formally introduced by A. Sen in the spirit of J. S. Mill's liberalism comprehension.

In addition, note that Arrow supposed that a social welfare function is defined for every possible combination of individual preferences, meaning that it must have a *universal domain.* Also, the condition called the *independence of irrelevant alternatives* is present, ensuring that "the way a society ranks a pair of alternative social states x and y should depend on the individual preferences only over *that* pair — in particular, *not* on how the other ('irrelevant') alternatives are ranked" (A. K. Sen 1995). These two conditions have a metalogical nature. We will use them as general properties of our system.

As a result of applying the abstraction operation to the original axioms of the Arrow–Sen theory, by ignoring the real nature of the symbols, we obtain dictatorship **D**, liberalism **L** and Pareto rule **P**, as the first–order predicate language formulae (B. Boričić 2007 and 2009), respectively, in the following form:

$$\mathbf{D} \qquad (\forall x, y)(xP_iy \to xPy), \text{ for some } i \ (1 \leq i \leq n)$$

$$\mathbf{L} \qquad (\exists x, y)(x \neq y \wedge (xP_iy \to xPy) \wedge (yP_ix \to yPx)), \text{ for each } i \ (1 \leq i \leq n),$$

$$\mathbf{P} \qquad (\forall x, y)(\bigwedge_{1 \leq i \leq n} xP_iy \to xPy)$$

as well as their logical negations, non–dictatorship **ND**, non–liberalism **NL** and non–Pareto rule **NP**, as follows:

$$\begin{array}{ll}
\textbf{ND} & \neg(\forall x,y)(xP_iy \rightarrow xPy),\ \text{for each } i\ (1 \leq i \leq n)\\
\textbf{NL} & (\forall x,y)(x = y \vee \neg(xP_iy \rightarrow xPy) \vee \neg(yP_ix \rightarrow yPx)),\\
 & \text{for some } i\ (1 \leq i \leq n),\\
\textbf{NP} & (\exists x,y)((\bigwedge_{1\leq i\leq n} xP_iy) \wedge \neg xPy)
\end{array}$$

In the sequel, these conditions will be 'translated' to the analogous 'almost propositional' formulae.

3. A Sketch of von Wright's Preference Logic

The logic of preference $\mathcal{W}(P)$, as introduced by G. H. von Wright (G. H. von Wright 1963 and 1972), is a formal theory based on the propositional language extended by a symbol P for a binary preference predicate such that 'APB' expresses that 'the alternative A is preferred to the alternative B', where A and B present propositional formulae. More formally, *the language* of $\mathcal{W}(P)$ consists of: (a) a denumerable set of symbols for propositional letters $p, q, r, ...$ (with or without subscripts), (b) symbols for the Boolean propositional connectives $\neg, \wedge, \vee, \rightarrow$ and $\leftrightarrow$ (for negation, conjunction, disjunction, implication and equivalence, respectively), (c) a symbol for a binary preference predicate P, and (d) parentheses.

The set of *alternatives* of $\mathcal{W}(P)$ is inductively defined as a set of propositional formulae over propositional letters and propositional connectives, i.e. the Boolean combinations of propositional letters. Propositional letters can be understood as metavariables for the elementary alternatives, and we will use capitals $A, B, C, \ldots$, with or without subscripts, as metavariables for complex combinations of elementary alternatives.

Note that in the case of classical two–valued propositional logic, the set of mutually non–equivalent alternatives built up over the list of m elementary alternatives (propositional letters) consists of 2^{2^m} elements.

Atomic formulae of $\mathcal{W}(P)$ are expressions of the form APB, where A and B are alternatives. *Formulae* of $\mathcal{W}(P)$ are the Boolean combinations of atomic formulae, i. e. expressions defined inductively over atomic formulae and the set of propositional connectives. We also accept the usual conventions regarding the use of parentheses.

The *logic of preference* $\mathcal{W}(P)$ is an extension of the classical propositional logic[2], over the set of formulae defined above, extended by the following basic axiom—schemata:

$$\text{(As)} \quad APB \to \neg(BPA)$$
$$\text{(Tr)} \quad APB \wedge BPC \to APC$$
$$\text{(Cnn)} \quad APB \to APC \vee CPB$$

and the following additional axioms defining some special properties of P:

$$\text{(w1)} \quad APB \leftrightarrow (A \wedge \neg B)P(\neg A \wedge B)$$
$$\text{(w2)} \quad A \vee BPC \leftrightarrow APC \wedge BPC$$
$$\text{(w3)} \quad APB \vee C \leftrightarrow APB \wedge APC$$
$$\text{(w4)} \quad APB \leftrightarrow ((A \wedge C)P(B \wedge C)) \wedge ((A \wedge \neg C)P(B \wedge \neg C))$$

meaning, for instance, that a conjunctive expansion of preference P is possible (by (w1)), and that the disjunctive preferences are 'conjunctively distributive' (by (w2) and (w3)).

It is not difficult to see that, for asymmetry (As), transitivity (Tr), and connectivity (Cnn), it holds $\text{(As)}, \text{(Cnn)} \vdash \text{(Tr)}$, where by '$\vdash$' we denote the classical deduction relation; but also that $\text{(As)}, \text{(Tr)} \nvdash$ (Cnn). This is the reason that the transitivity condition could be omitted from the list of axioms given above. Note that, in this case, P is an irreflexive relation, i.e., for each A, $\neg(APA)$ holds.

A preference relation defined in this way, satisfying axioms (As) and (Cnn), is usually called a *strict preference relation.* By means of this

[2]An interesting alternative approach could be based on some non–classical propositional logics.

strict preference relation P we can define a *relation of indifference* I as follows:

$$AIB \ \text{ iff(def) } \ \neg(APB) \wedge \neg(BPA)$$

It is provable that I is a reflexive, symmetric, and transitive relation, i.e., I is an equivalence relation. Also, it can be shown that:

$$APB \wedge BIC \vdash APC$$
$$APB \wedge AIC \vdash CPB$$

A *weak preference relation* R is usually defined by means of a strict preference and an indifference relation in the following way:

$$ARB \ \text{ iff(def) } \ (APB) \vee (AIB)$$

It is provable that R is a linear (consequently, reflexive also) and transitive relation, meaning that R satisfies *rational choice axioms* (K. Arrow 1951). The following statements are also provable:

$$APB \wedge BRC \vdash APC$$
$$ARB \wedge BPC \vdash APC$$

4. Combining von Wright's Preference Logics

In this section we present a construction of combined von Wright's preference logics, resulting a natural tool to express some problems of generating social preference relations by personal relations.

Let $\mathcal{W}(P)$ and $\mathcal{W}(Q)$ be two logics of preference. By $\mathcal{W}(P,Q)$ we denote the deductive closure of the union $\mathcal{W}(P) \cup \mathcal{W}(Q)$ of all provable formulae in $\mathcal{W}(P)$ or $\mathcal{W}(Q)$, i.e., minimal extension of $\mathcal{W}(P) \cup \mathcal{W}(Q)$ closed for *modus ponens,* where P and Q are symbols for the basic binary preference predicates of $\mathcal{W}(P)$ and $\mathcal{W}(Q)$, respectively.

Lemma. *The logic $\mathcal{W}(P,Q)$ is a conservative extension of $\mathcal{W}(Q)$.*

Proof. We use the induction on the length of proof in $\mathcal{W}(P,Q)$. Let us denote by $\Gamma, \Delta, \ldots$ formulae of $\mathcal{W}(P,Q)$, and by $\Gamma(P/Q)$ a formula obtained by substitution of each occurrence of binary preference relation symbol P by binary preference relation symbol Q in Γ. For each axiom Γ of $\mathcal{W}(P, Q)$, obviously, $\Gamma(P/Q)$ is an axiom of $\mathcal{W}(Q)$. Also, bearing in mind that $(\Gamma \rightarrow \Delta)(P/Q) = \Gamma(P/Q) \rightarrow \Delta(P/Q)$, from $\Gamma(P/Q)$ and $(\Gamma \rightarrow \Delta)(P/Q)$, we can infer $\Delta(P/Q)$ in $\mathcal{W}(Q)$, where, by induction hypothesis, both $\Gamma(P/Q)$ and $(\Gamma \rightarrow \Delta)(P/Q)$ are provable in $\mathcal{W}(Q)$. This means that, for each formula Γ, if Γ is provable in $\mathcal{W}(P, Q)$, then Γ is provable in $\mathcal{W}(Q)$.$\square$

Consequently, $\mathcal{W}(P, Q)$ can be considered a regular result of combining $\mathcal{W}(P)$ and $\mathcal{W}(Q)$.

This process may be extended. We introduce *the combined von Wright's preference logic,* denoted by $\mathcal{W}(P, P_1, \ldots, P_n)$. Let us extend the language of $\mathcal{W}(P)$ by a finite list of symbols for the binary preference predicates $P_1, \ldots, P_n$, for $n \geq 2$. The *atomic formulae* of $\mathcal{W}(P, P_1, \ldots, P_n)$ are expressions of the form APB and AP_iB $(1 \leq i \leq n)$, where A and B are alternatives. *Formulae* of $\mathcal{W}(P, P_1, \ldots, P_n)$ are the Boolean combinations of atomic formulae.

For instance, the formulae of $\mathcal{W}(P, P_1, \ldots, P_n)$ will appear as follows: $AP_1B \rightarrow BP_nC$, $BPA \vee \neg(AP_1B)$,... The intended meaning of AP_iB that 'the i–th person prefers alternative A to alternative B', and APB means that 'the society prefers A to B'.

By *combined von Wright's preference logic* we mean a simple deductive closure of the union of the following von Wright preference logics $\mathcal{W}(P)$ and $\mathcal{W}(P_i)$ $(1 \leq i \leq n)$, i.e., the set of all consequences of the union of logics $\mathcal{W}(P)$ and $\mathcal{W}(P_i)$ $(1 \leq i \leq n)$, which is, obviously, closed for *modus ponens.*

Based on the Lemma above, by induction on n, we can prove that $\mathcal{W}(P, P_1, \ldots, P_n)$ is a conservative extension of $\mathcal{W}(P)$ and of $\mathcal{W}(P_i)$, for each i $(1 \leq i \leq n)$, meaning that $\mathcal{W}(P, P_1, \ldots, P_n)$ can be treated as a result of combining $\mathcal{W}(P)$, $\mathcal{W}(P_1)$,..., $\mathcal{W}(P_n)$.

5. Some Extensions of the Combined von Wright's Preference Logic

In this part we propose the propositional counterparts of dictatorship

D, liberalism **L**, and the Pareto rule **P**, respectively, in the following form:

D: there exists i $(1 \leq i \leq n)$, such that, for any A and B,

$$AP_iB \to APB,$$

L: for each i $(1 \leq i \leq n)$, there exist two mutually non–equivalent alternatives A and B, such that

$$(AP_iB \to APB) \wedge (BP_iA \to BPA)$$

P: $\bigwedge_{1\leq i\leq n} AP_iB \to APB,$

as well as their logical negations, non–dictatorship **ND**, non–liberalism **NL** and non–Pareto rule **NP**, as follows:

ND : for each i $(1 \leq i \leq n)$, there exist two alternatives A and B such that

$$AP_iB \wedge \neg APB$$

NL : for some i $(1 \leq i \leq n)$ and each two mutually non–equivalent A and B,

$$(AP_iB \wedge \neg(APB)) \vee (BP_iA \wedge \neg(BPA))$$

NP there exist A and B such that

$$\left(\left(\bigwedge_{1\leq i\leq n} AP_iB\right) \wedge \neg APB\right)$$

These axioms, expressed as formulae of the extended propositional language on which von Wright's preference logic is based, enable us to make a pure formal logical analysis of some relationships between the conditions under consideration.

Let $V = \{1, 2, \ldots, n\}$ be the set of all individuals. A subset $G \subseteq V$ will be called a *decisive group* over the pair of two mutually non–equivalent alternatives A and B (A. K. Sen 1995), if

$$(\forall i \in G)AP_iB \to APB \quad \text{and} \quad (\forall i \in G)BP_iA \to BPA$$

The group G decisive over all pairs of alternatives will be called *the decisive group.* Consequently, the liberalism axiom **L** may be formulated as follows: each individual presents a decisive group over at least one pair of mutually non–equivalent alternatives.

We also want our system $\mathcal{W}(P, P_1, \ldots, P_n)$ to work for each number of individuals $n \geq 2$ and for every possible combination of individual preferences. This condition corresponds to the *universal domain* **U**. One more condition we want to satisfy is the *independence of irrelevant alternatives* **IIA**, providing that the way a society decides APB, for a pair of alternatives A and B, should depend on the individual preferences over only that pair, and not on how the other 'irrelevant' alternatives are treated by individuals or society. These two conditions have metatheoretical significance with respect to $\mathcal{W}(P, P_1, \ldots, P_n)$, and both of them can be considered as rules of process of assigning a social preference relation P to any n–tuple $(P_1, \ldots, P_n)$ of individual preferences in the context of combined von Wright's preference logic $\mathcal{W}(P, P_1, \ldots, P_n)$.

6. Impossibility Theorems in Combined von Wright's Preference Logic

Let us first present a propositional counterpart of Sen's famous result (A. K. Sen 1970a or 1970b) known as the 'impossibility of Paretian liberal' or the 'liberal paradox':

Theorem. *The procedure of assigning a social preference relation to a finite number of individual preference relations over a finite set of possible alternatives satisfying condition* **U**, *based on an asymmetric and transitive subsystem of combined von Wright's preference logic, including axioms* **L** *and* **P**, *does not exist.*

Proof. First we note that an immediate consequence of the asymmetry axiom is that P is an irreflexive relation: $\neg(APA)$. Let us suppose that the set of all individuals consists of $n(\geq 2)$ persons and that, for the alternatives $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$, where A_i and B_i are mutually non–equivalent, the particular cases of liberalism axiom

$$A_i P_i B_i \rightarrow A_i P B_i$$

hold, for all i $(1 \leq i \leq n)$, wherefrom we infer

$$A_1 P B_1 \wedge A_2 P B_2 \wedge \cdots \wedge A_n P B_n$$

Now, we analyze the preferences of the first two individuals only, bearing in mind that, for alternatives A_1, B_1, A_2 and B_2, $A_1 P B_1 \wedge A_2 P B_2$ holds. In the case when B_1 is equivalent to A_2 or A_1 is equivalent to B_2, it is possible to suppose that $B_2 P_i A_1$, for all i $(1 \leq i \leq n)$, or $B_1 P_i A_2$, for all i $(1 \leq i \leq n)$, respectively, from which, by the Pareto rule, we can infer $B_2 P A_1$ or $B_1 P A_2$. So, if $B_1 \leftrightarrow A_2$, then, from $A_1 P B_1 \wedge A_2 P B_2 \wedge B_2 P A_1$, by (Tr), we conclude $A_1 P A_1$. Similarly, if $A_1 \leftrightarrow B_2$, then, from $A_1 P B_1 \wedge B_1 P A_2 \wedge A_2 P B_2$, again, by (Tr), we conclude $A_1 P A_1$. Finally, if $\neg(B_1 \leftrightarrow A_2) \wedge \neg(A_1 \leftrightarrow B_2)$, then it is possible to suppose that, for all i $(1 \leq i \leq n)$, $B_2 P_i A_1 \wedge B_1 P_i A_2$, from which, by the Pareto rule, we can infer $B_2 P A_1 \wedge B_1 P A_2$, and then, from $A_1 P B_1 \wedge B_1 P A_2 \wedge A_2 P B_2 \wedge B_2 P A_1$, by (Tr), we conclude $A_1 P A_1$, violating that P is irreflexive, in each case.$\square$

Note that, essentially, the presented proof is in the spirit of Sen's original proof (A. K. Sen 1970b or 1995), based on the minimal liberalism argument, meaning that there are at least two persons decisive over two existing pairs of distinct alternatives.

A counterpart of Arrow's general impossibility theorem can be formulated as follows:

Theorem. *A procedure of assigning a social preference relation to a finite number of individual preference relations over a finite set of possible alternatives satisfying conditions* **U** *and* **IIA**, *based on an asymmetric and transitive subsystem of combined von Wright's preference logic, including axioms* **ND** *and* **P**, *does not exist.*

The basic idea followed here, presented by Sen (A. K. Sen 1995), in the context of the Arrow–Sen theory, is to divide the proof into two parts by showing 'the Field–Expansion Lemma': *if a group is* *decisive over any pair of alternatives, it is decisive,* and 'the Group–Contraction Lemma': *if a group is decisive, then so is some smaller group contained in it.*

Proof. Let A, B, C and D be four alternatives, all mutually non–equivalent, and let G be a decisive group over A and B. Let $\forall i(i \in G \rightarrow CP_iA \wedge AP_iB \wedge BP_iD)$ and $\forall i(i \notin G \rightarrow CP_iA \wedge BP_iD)$, which is possible by unrestricted domain. As G is decisive over A and B, we can conclude APB. On the other hand, by the Pareto rule, we can infer CPA and BPD. Finally, from CPA, APB, and BPD, by transitivity, we infer CPD. If this conclusion is influenced by individual preferences over any pair other than C and D, then the condition of independence of irrelevant alternatives is violated. Thus, C must be ranked above D simply by virtue of everyone in G preferring C to D (since others can have any preference whatsoever over this pair). Consequently, G is decisive over C and D, meaning that *if a group is decisive over any pair of alternatives, it is decisive.* The proof when the alternatives are not all mutually non–equivalent is similar. Let G be a decisive group and E and F its partition, meaning that E and F are non–empty, $E \cup F = G$ and $E \cap F = \emptyset$. Let $\forall i(i \in E \rightarrow AP_iB \wedge AP_iC)$ and $\forall i(i \in F \rightarrow AP_iB \wedge CP_iB)$. Now, as by bisection method, we have the two following possibilities: APC, when E is decisive over A and C; and if E is not decisive over A and C, then CRA, i.e., C is at least as good as A. As G is decisive over A and B, and $CRA \wedge APB \rightarrow CPB$, we infer CPB. But, only for members $i \in F$, we have CP_iB, meaning that, by the first part of the proof, F is decisive over C and B. Consequently, either E or F must be a decisive group, which means that *if a group (of more than one person) is decisive, then so is some smaller group contained within it.* Finally, by the Pareto rule, the group of all individuals is decisive. Since it is finite, by successive partitioning, each time picking its decisive part, we arrive at a decisive individual, a dictator.$\square$

Let us now formulate some facts regarding the logical interdependencies of the axioms under consideration appearing in this context as well (B. Boričić 2009 and G. Chichilnisky 1982):

Lemma. *In each subsystem of combined von Wright's preference logic, if* **D** *is provable, then* **P** *is provable.*

Proof. This is the result of a pure formal deduction of **P** from **D**, by weakening the antecedent of the formula $AP_iB \rightarrow APB$ n–times.□

Consequently, if **NP**, then **ND**. This means that each dictatorial society is Paretian, or, equivalently, that each non–Paretain society is non–dictatorial.

As an immediate consequence of the previous Lemma and the 'liberal paradox', we have:

Corollary. *In each asymmetric and transitive subsystem of combined von Wright's preference logic including axiom* **U**, *if* **L** *is provable, then* **ND** *is provable.*

In terms of 'possible' (consistent) and 'impossible' (inconsistent) combinations of axioms of combined von Wright's preference logic, we can conclude that if the system containing **D** is possible, then the system obtained by substituting **D** by **P** is possible; if the system containing **D** is possible, then its extension by **P** is possible; if the system containing **P** is impossible, then the system obtained by substituting **P** by **D** is impossible; if the system containing **P** is impossible, then its extension by **D** is impossible; if the system containing **NP** is possible, then the system obtained by substituting **NP** by **ND** is possible; if the system containing **NP** is possible, then its extension by **ND** is possible; if the system containing **ND** is impossible, then the system obtained by substituting **NP** by **ND** is impossible; if the system containing **ND** is impossible, then its extension by **ND** is impossible; the system containing **D** and **NP** is impossible ('the impossibility of a non–Paretian dictator'); the system containing **L**, **D**, and **NP** is impossible; the system containing **NL**, **D**, and **NP** is impossible. Consequently, if **D** is provable, then **NL** is provable. This means that each liberal society is non–dictatorial, or, equivalently, that each dictatorial society is non–liberal. It also means that, under the conditions of Arrow's Impossibility Theorem, **P** and **D** are mutually equivalent. This result was obtained by Chichilnisky

(G. Chichilnisky 1982) in a topological context. All these facts are simple and useful examples of interdependence and 'possibility' and 'impossibility' results suitable for presentation to students.

7. Concluding Remarks

In contrast to the traditional approach to social choice theory (K. Arrow 1951 and A. K. Sen 1970b) and preference logic (G. H. von Wright 1963 and 1972) we propose an interpretation of Arrow's and Sen's impossibility theorems in the context of combined von Wright's preference logic. The original Arrow–Sen theory, although based on principles of preference logic, does not provide the possibility of constructing complex alternatives, while, on the other hand, von Wright's preference logic does not provide the possibility of simultaneously expressing individual and social preferences. We believe that this interpretation, based on the recently introduced and developed combining logics method (D. M. Gabbay 1999), overcomes these defects and, due to its simplicity in this case, enables us to present these profound results and ideas to a wider circle of researchers and readers. Note that we did not use the axioms (w1)–(w4) of $\mathcal{W}(P)$ in the proofs given in part 6 of this paper. This means that it is possible to prove the presented results in a narrower framework. Namely, the formulation of the theorems define the minimal context in which these famous statements are provable.

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